Exercise 2.1

Find the principal values of the following:

1. $\sin^{-1}\left(-\frac{1}{2}\right)$

2. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

3. Cosec $^{-1}$ **(2)**

4. $\tan^{-1}\left(-\sqrt{3}\right)$

5. $\cos^{-1}\left(\frac{1}{2}\right)$

6. $\tan^{-1}(-1)$

7. $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

8. $\cot^{-1}\left(\sqrt{3}\right)$

9. $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$

10. $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$

Solution 1: Consider y = $\sin^{-1}\left(-\frac{1}{2}\right)$
Solve the above equation, we have
sin y = -1/2
We know that sin π/6 = ½
So, sin y = - sin π/6

$$\sin y = \sin\left(-\frac{\pi}{6}\right)$$

Since range of principle value of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Principle value of $\sin^{-1}\left(-\frac{1}{2}\right)$ is - π/6.

Solution 2:

Let y = $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Cos y = cos π/6 (as cos π/6 = $\sqrt{3}$ / 2)

y = π/6

Since range of principle value of $\cos^{-1}$ is [0, π]

Therefore, Principle value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is π/6

Solution 3: Cosec $^{-1}$ (2)

Let y = Cosec $^{-1}$ (2)

Cosec y = 2

We know that, cosec π /6 = 2

So Cosec y = cosec π /6

Since range of principle value of cosec^{-1} is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Therefore, Principle value of Cosec $^{-1}$ (2) is Π/6.

Solution 4: $\tan^{-1}\left(-\sqrt{3}\right)$

Let y = $\tan^{-1}\left(-\sqrt{3}\right)$

tan y = - tan π/3

or tan y = tan (-π/3)

Since range of principle value of $\tan^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Therefore, Principle value of $\tan^{-1}\left(-\sqrt{3}\right)$ is – π/3.

Solution 5: $\cos^{-1}\left(\frac{-1}{2}\right)$

y = $\cos^{-1}\left(\frac{-1}{2}\right)$

cos y = -1/2

$\cos y = -\cos\frac{\pi}{3}$

cos y = cos(π - π/3) = cos (2π/3)

Since principle value of $\cos^{-1}$ is [0, π]

Therefore, Principle value of $\cos^{-1}\left(\frac{-1}{2}\right)$ is 2π/3.

Solution 6: $\tan^{-1}(-1)$

Let y = $\tan^{-1}(-1)$

tan (y) = -1

tan y = -tan π/4

$\tan y = \tan\left(-\frac{\pi}{4}\right)$

Since principle value of $\tan^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Therefore, Principle value of $\tan^{-1}(-1)$ is - π/4.

Solution 7: $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

y = $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

sec y = 2/$\sqrt{3}$

$\sec y = \sec\frac{\pi}{6}$

Since principle value of $\sec^{-1}$ is [0, п]

Therefore, Principle value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ is п/6

Solution 8: $\cot^{-1}\left(\sqrt{3}\right)$

y = $\cot^{-1}\left(\sqrt{3}\right)$

cot y = √3

cot y = п/6

Since principle value of $\cot^{-1}$ is [0, п]

Therefore, Principle value of $\cot^{-1}\left(\sqrt{3}\right)$ is п/6.

Solution 9: $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$

Let y = $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$

$\cos y = -\frac{1}{\sqrt{2}}$

$\cos y = -\cos\frac{\pi}{4}$

$\cos y = \cos\left(\pi - \frac{\pi}{4}\right) = \cos\frac{3\pi}{4}$

Since principle value of $\cos^{-1}$ is [0, π]

Therefore, Principle value of $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$ is 3 π/ 4.

Solution 10. $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$

Ley y = $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$

$\operatorname{cosec} y = -\sqrt{2}$

$\operatorname{cosec} y = \operatorname{cosec}\frac{-\pi}{4}$

Since principle value of $\operatorname{cosec}^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Therefore, Principle value of $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$ is – π/4

Find the values of the following:

11. $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$

12. $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$

13. If $\sin^{-1} x = y$, then

(A) $0 \le y \le \pi$

(B) $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(C) $0 < y < \pi$

(D) $-\frac{\pi}{2} < y < \frac{\pi}{2}$

14. $\tan^{-1}(\sqrt{3}) - \sec^{-1}(-2)$ is equal to

(A) π

(B) - π/3

(C) π/3

(D) 2 π/3

Solution 11. $\tan^{-1}(1) + \cos^{-1}\left(\frac{-1}{2}\right) + \sin^{-1}\left(\frac{-1}{2}\right)$

$$= \tan^{-1}\tan\frac{\pi}{4} + \cos^{-1}\left(-\cos\frac{\pi}{3}\right) + \sin^{-1}\left(-\sin\frac{\pi}{6}\right)$$

$$= \frac{\pi}{4} + \cos\left(\pi - \frac{\pi}{3}\right) + \sin^{-1}\sin\left(-\frac{\pi}{6}\right)$$

$$= \frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6}$$

$$= \frac{3\pi + 8\pi - 2\pi}{12}$$

$$= \frac{9\pi}{12} = \frac{3\pi}{4}$$

Solution 12:

Let $\cos^{-1}\left(\frac{1}{2}\right) = x$. Then, $\cos x = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right)$

$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

Let $\sin^{-1}\left(\frac{1}{2}\right) = y$. Then, $\sin y = \frac{1}{2} = \sin\left(\frac{\pi}{6}\right)$

$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

Now,

$$\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} + \frac{2\pi}{6}$$

$$= \frac{\pi}{3} + \frac{\pi}{3}$$

$$= \frac{2\pi}{3}$$

Solution 13: Option (B) is correct.

Given $\sin^{-1} x = y$,

The range of the principle value of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Therefore, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

Solution 14:

Option (B) is correct.

$\tan^{-1}(\sqrt{3}) - \sec^{-1}(-2) = \tan^{-1}(\tan \pi/3) - \sec^{-1}(-\sec \pi/3)$

$= \pi/3 - \sec^{-1}(\sec(\pi - \pi/3))$

$= \pi/3 - 2\pi/3 = -\pi/3$

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Prove the following

1.

$3\sin^{-1}x = \sin^{-1}(3x - 4x^3), x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

Solution:

$3\sin^{-1}x = \sin^{-1}(3x - 4x^3), x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

(Use identity: $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$)

Let $x = \sin\theta$, then

$\theta = \sin^{-1}x$

Now, RHS

$= \sin^{-1}(3x - 4x^3)$

$= \sin^{-1}(3\sin\theta - 4\sin^3\theta)$

$= \sin^{-1}(\sin 3\theta)$

$= 3\theta$

$= 3\sin^{-1}x$

= LHS

Hence Proved

2.

$3\cos^{-1}x = \cos^{-1}(4x^3 - 3x), x \in \left[\frac{1}{2}, 1\right]$

Solution:

$$3\cos^{-1}x = \cos^{-1}\left(4x^3 - 3x\right), x \in \left[\frac{1}{2}, 1\right]$$

Using identity: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

Put x = cos θ

θ = $\cos^{-1}$ (x)

Therefore, cos 3 θ = $4x^3$ - 3x

RHS:

$\cos^{-1}\left(4x^3 - 3x\right)$

= $\cos^{-1}$ (cos 3 θ)

= 3 θ

= 3 $\cos^{-1}$ (x)

= LHS

Hence Proved.

3.

$$\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$$

Solution:

$$\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$$

Using identity: $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$

LHS = $\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}$

$= \tan^{-1}\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}\times\frac{7}{24}}$

$= \tan^{-1}\frac{48+77}{264-14}$

= tan $^{-1}$ (125/250)

= tan $^{-1}$ (1/2)

= RHS

Hence Proved

4.

$2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{31}{17}$

Solution:

Use identity: $2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}$.

LHS

= $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}$

= $\tan^{-1}\frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}+\tan^{-1}\frac{1}{7}$

= $\tan^{-1}(4/3) + \tan^{-1}(1/7)$

Again using identity:

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$$

We have,

$$\tan^{-1} \frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\times\frac{1}{7}}$$

= $tan^{-1}(\frac{28+3}{21-4})$

= $\tan^{-1}(31/17)$

RHS

Write the following functions in the simplest form:

5. $\tan^{-1} \frac{\sqrt{1+x^2}-1}{x}, x \neq 0$

Solution:

Let's say x = tan θ then θ = $\tan^{-1} x$

We get,

$$\tan^{-1} \frac{\sqrt{1+x^2}-1}{x} = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$= \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right)$

$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$

$= \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$

$= \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$

This is simplest form of the function.

6. $\tan^{-1}\frac{1}{\sqrt{x^2-1}}, |x| > 1$

Solution:

Let us consider, x = sec θ, then θ = $\sec^{-1}$ x

$\tan^{-1}\frac{1}{\sqrt{x^2-1}} = \tan^{-1}\frac{1}{\sqrt{\sec^2\theta - 1}}$

$= \tan^{-1}\frac{1}{\sqrt{\tan^2\theta}}$

$= \tan^{-1}\left(\frac{1}{\tan\theta}\right)$

= $\tan^{-1}$(cot θ)

= $\tan^{-1}$ tan(π/2 - θ)

= (π/2 - θ)

= π/2 – $\sec^{-1}$ x

This is simplest form of the given function.

7. $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$, $0 < x < \pi$

Solution:

$$\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right) = \tan^{-1}\left(\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}\right)$$

$$= \tan^{-1}\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\tan\frac{x}{2}\right) = \frac{x}{2}$$

8. $\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right), \frac{-\pi}{4} < x < \frac{3\pi}{4}$

Solution:

Divide numerator and denominator by cos x, we have

$$\tan^{-1}\left(\frac{\frac{cox(x)}{\cos(x)} - \frac{\sin(x)}{\cos(x)}}{\frac{\cos(x)}{\cos(x)} + \frac{\sin(x)}{\cos(x)}}\right)$$

$$= \tan^{-1}\left(\frac{1 - \frac{\sin(x)}{\cos(x)}}{1 + \frac{\sin(x)}{\cos(x)}}\right)$$

$$= \tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right)$$

$$\tan^{-1}\left(\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right)$$

= tan $^{-1}$ tan(π/4 – x)

= π/4 – x

9. $\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}, |x| < a$

Solution:

Put x = a sin θ, which implies sin θ = x/a and θ = $\sin^{-1}$(x/a)

Substitute the values into given function, we get

$$\tan^{-1}\frac{x}{\sqrt{a^2-x^2}} = \tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2-a^2\sin^2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{a\sin\theta}{a\sqrt{1-\sin^2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta}\right)$$

= $\tan^{-1}$(tan θ)

= θ

= $\sin^{-1}$(x/a)

10. $\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right), a > 0; \frac{-a}{\sqrt{3}} < x < \frac{a}{\sqrt{3}}$

Solution:

After dividing numerator and denominator by a^3 we have

$$\tan^{-1}\left(\frac{3\left(\frac{x}{a}\right)-\left(\frac{x}{a}\right)^3}{1-3\left(\frac{x}{a}\right)^2}\right)$$

Put x/a = tan θ and θ = $\tan^{-1}$(x/a)

$$= \tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$$

= $\tan^{-1}$ (tan 3 θ)

= 3 θ

= 3 $\tan^{-1}$(x/a)

Find the values of each of the following:

11. $\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]$

Solution:

$$= \tan^{-1}\left[2\cos\left(2\sin^{-1}\sin\frac{\pi}{6}\right)\right]$$

$$= \tan^{-1}\left[2\cos\left(2\times\frac{\pi}{6}\right)\right]$$

= $\tan^{-1}$ (2 cos π/3)

= $\tan^{-1}$(2 x ½)

= $\tan^{-1}$ (1)

= $\tan^{-1}$ (tan (π/4))

= π/4

12. cot ($\tan^{-1}a + \cot^{-1}a$)

Solution:

cot ($\tan^{-1}a + \cot^{-1}a$) = cot π/2 = 0

Using identity: $\tan^{-1}a + \cot^{-1}a = \pi/2$

13.

$$\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right], |x|<1, y>0 \text{ and } xy<1$$

Solution:

Put x = tan θ and y = tan Φ, we have

$$\tan\frac{1}{2}\left[\sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta}+\cos^{-1}\frac{1-\tan^2\phi}{1+\tan^2\phi}\right]$$

= tan1/2[$\sin^{-1}$ sin 2 θ + $\cos^{-1}$ cos 2 Φ]

= tan (1/2) [2 θ + 2 Φ]

= tan (θ + Φ)

$$= \frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$$

= (x+y) / (1-xy)

14. If $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$**, then find the value of x.**

Solution:

We know that, sin 90 degrees = sin π/2 = 1

So, given equation turned as,

$$\sin^{-1}\frac{1}{5}+\cos^{-1}x=\frac{\pi}{2}$$

$$\cos^{-1}x=\frac{\pi}{2}-\sin^{-1}\frac{1}{5}$$

Using identity: $\sin^{-1} t + \cos^{-1} t$ = π/2

We have, $\cos^{-1} x = \cos^{-1} \frac{1}{5}$

Which implies, the value of x is 1/5.

15. If $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$**, then find the value of x.**

Solution:

We have reduced the given equation using below identity:

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}$$

$$\tan^{-1}\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)} = \frac{\pi}{4}$$

or

$$\tan^{-1}\frac{(x-1)(x+2)+(x+1)(x-2)}{(x-2)(x+2)-(x-1)(x+1)} = \frac{\pi}{4}$$

or

$$\tan^{-1}\frac{x^2+2x-x-2+x^2-2x+x-2}{x^2-4-\left(x^2-1\right)} = \frac{\pi}{4}$$

$or \ \frac{2x^2-4}{x^2-4-x^2+1} = \tan\left(\frac{\pi}{4}\right)$

or (2x^2 - 4)/-3 = 1

or 2x^2 = 1

or x = ± $\frac{1}{\sqrt{2}}$

The value of x is either $\frac{1}{\sqrt{2}}\ or -\frac{1}{\sqrt{2}}$

Find the values of each of the expressions in Exercises 16 to 18.

16. $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$

Solution:

Given expression is $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$

First split $\frac{2\pi}{3}$ as $\frac{(3\pi-\pi)}{3}$ or $\pi - \frac{\pi}{3}$

After substituting in given we get,

$\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \sin^{-1}\left(\sin\left(\pi - \frac{\pi}{3}\right)\right) = \frac{\pi}{3}$

Therefore, the value of $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$ is $\frac{\pi}{3}$

17. $tan^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right)$

Solution:

Given expression is $tan^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right)$

First split $\frac{3\pi}{4}$ as $\frac{(4\pi-\pi)}{4}$ or $\pi - \frac{\pi}{4}$

After substituting in given we get,

$tan^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right) = \tan^{-1}\left(\tan\left(\pi - \frac{\pi}{4}\right)\right) = -\frac{\pi}{4}$

The value of $tan^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right)$ is $\frac{-\pi}{4}$.

18. $\tan\left(sin^{-1}\left(\frac{3}{5}\right) + \cot^{-1}\frac{3}{2}\right)$

Solution:

Given expression is $\tan\left(sin^{-1}\left(\frac{3}{5}\right) + \cot^{-1}\frac{3}{2}\right)$

Putting, $sin^{-1}\left(\frac{3}{5}\right) = x \; and \; \cot^{-1}\left(\frac{3}{2}\right) = y$

Or sin(x) = 3/5 and cot y = 3/2

Now, sin(x) = 3/5 => cos x = $\sqrt{1-\sin^2 x}$ = 4/5 and sec x = 5/4

(using identities: cos x = $\sqrt{1-\sin^2 x}$ and sec x = 1/cos x)

Again, tan x = $\sqrt{\sec^2 x - 1}$ = $\sqrt{\frac{25}{16} - 1}$ = ¾ and tan y = 1/cot(y) = 2/3

Now, we can write given expression as,

$\tan(sin^{-1}\left(\frac{3}{5}\right) + \cot^{-1}\frac{3}{2})$ = tan(x + y)

$= \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{\frac{3}{4} + \frac{2}{3}}{1 - \frac{3}{4} \times \frac{2}{3}}$

= 17/6

19. $\cos^{-1}(\cos \frac{7\pi}{6})$ is equal to

(A) 7π/6 (B) 5 π/6 (C) π/3 (D) π/6

Solution:

Option (B) is correct.

Explanation:

$\cos^{-1}(\cos \frac{7\pi}{6})$ = $\cos^{-1}(\cos (2\pi - \frac{7\pi}{6})$

(As cos (2 π − A) = cos A)

Now $2\pi - \frac{7\pi}{6} = \frac{12\pi - 7\pi}{6} = \frac{5\pi}{6}$

20. $\sin\left[\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right]$ **is equal to**

(A) ½ (B) 1/3 (C) ¼ (D) 1

Solution:

Option (D) is correct

Explanation:

First solve for: $\sin^{-1}(-\frac{1}{2})$

$$\sin^{-1}\left(-\frac{1}{2}\right)=\sin^{-1}\left(-\sin\frac{\pi}{6}\right)=\sin^{-1}\left[\sin\left(-\frac{\pi}{6}\right)\right]$$

$= -\pi/6$

Again,

$$\sin\left[\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right]$$

$$=\sin\left[\frac{\pi}{3}-\left(-\frac{\pi}{6}\right)\right]$$

$$=\sin\left[\frac{\pi}{3}+\frac{\pi}{6}\right]$$

$= \sin(\pi/2)$

$= 1$

21. $\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3})$ is equal to

(A) π (B) -π/2 (C) 0 (D) $2\sqrt{3}$

Solution:

Option (B) is correct.

Explanation:

$\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3})$ can be written as

$$= \tan^{-1}\tan\frac{\pi}{3} - \cot^{-1}\left(-\cot\frac{\pi}{6}\right)$$

$$= \frac{\pi}{3} - \cot^{-1}\left[\cot\left(\pi - \frac{\pi}{6}\right)\right]$$

$$= \frac{\pi}{3} - (\pi - \frac{\pi}{6})$$

$$= \frac{\pi}{3} - \frac{5\pi}{6}$$

$$= \frac{-3\pi}{6}$$

$= -\pi/2$

Miscellaneous Exercise

Find the value of the following:

1. $\cos^{-1}(\cos\frac{13\pi}{6})$

Solution:

First solve for, $\cos\frac{13\pi}{6} = \cos(2\pi + \frac{\pi}{6}) = \cos\frac{\pi}{6}$

Now: $\cos^{-1}(\cos\frac{13\pi}{6}) = \cos^{-1}(\cos\frac{\pi}{6}) = \frac{\pi}{6} \in [0, \pi]$

[As $\cos^{-1}\cos(x) = x$ if $x \in [0, \pi]$]

So the value of $\cos^{-1}(\cos\frac{13\pi}{6})$ is $\frac{\pi}{6}$

2. $tan^{-1}(\tan\frac{7\pi}{6})$

Solution:

First solve for, $\tan\frac{7\pi}{6} = \tan(\pi + \frac{\pi}{6}) = \tan\frac{\pi}{6}$

Now: $tan^{-1}(\tan\frac{7\pi}{6}) = \tan^{-1}(\tan\frac{\pi}{6}) = \frac{\pi}{6} \in (-\pi/2, \pi/2)$

[As $\tan^{-1}\tan(x) = x$ if $x \in (-\pi/2, \pi/2)$]

So the value of $tan^{-1}(\tan\frac{7\pi}{6})$ is $\frac{\pi}{6}$

3. Prove that $2\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{24}{7}$

Solution:

Step 1: Find the value of cos x and tan x

Let us consider $\sin^{-1}\frac{3}{5} = x$, then sin x = 3/5

So, cos x = $\sqrt{1-\sin^2 x} = \sqrt{1-(\frac{3}{5})^2}$ = 4/5 tan x = sin x/ cos x = ¾

Therefore, x = $\tan^{-1}$ (3/4), substitute the value of x,

⇨ $\sin^{-1}\frac{3}{5} = \tan^{-1}(\frac{3}{4})$ …..(1)

Step 2: Solve LHS

$2\sin^{-1}\frac{3}{5} = 2\tan^{-1}\frac{3}{4}$

Using identity: $2\tan^{-1} x = \tan^{-1} = \tan^{-1}(\frac{2x}{1-x^2})$, we get

= $\tan^{-1}(\frac{2(\frac{3}{4})}{1-(\frac{3}{4})^2})$

= $\tan^{-1}(24/7)$

= RHS

Hence Proved.

4. Prove that $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{77}{36}$

Solution:

Let $\sin^{-1}(\frac{8}{17}) = x$ then sin x = 8/17

Again, cos x = $\sqrt{1-\sin^2 x}$ = $\sqrt{1-\frac{64}{289}}$ = 15/17

And tan x = sin x / cos x = 8/ 15

Again,

Let $\sin^{-1}(\frac{3}{5}) = y$ then sin y = 3/5

Again, cos y = $\sqrt{1-\sin^2 y}$ = $\sqrt{1-\frac{9}{25}}$ = 4/5

And tan y = sin y / cos y = ¾

Solve for tan(x + y), using below identity,

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$= \frac{\frac{8}{15}+\frac{3}{4}}{1-\frac{8}{15}\times\frac{3}{4}}$$

$= \frac{32+45}{60-24}$

= 77/36

This implies x + y = $\tan^{-1}$(77/36)

Substituting the values back, we have

$\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{77}{36}$ (Proved)

5. Prove that $\cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)$

Solution:

Let $\cos^{-1}\frac{4}{5} = \theta$

$$\cos\theta = \frac{4}{5}$$

$$\sin\theta = \sqrt{1-\cos^2\theta}$$

$$= \sqrt{1-\frac{16}{25}}$$

$$= \frac{3}{5}$$

Let $\cos^{-1}\frac{12}{13} = \phi$

$$\cos\phi = \frac{12}{13}$$

$$\sin\phi = \sqrt{1-\cos^2\phi}$$

$$= \sqrt{1-\frac{144}{169}}$$

$$= \frac{5}{13}$$

Solve the expression, Using identity: cos (θ + ϕ) = cosθ cos ϕ - sinθ sin ϕ

= 4/5 x 12/13 - 3/5 x 5/13

= (48-15)/65

= 33/65

This implies cos (θ + ϕ) = 33/65

or θ + ϕ = $\cos^{-1}$ (33/65)

Putting back the value of θ and ϕ, we get

$$\cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)$$

Hence Proved.

6. Prove that $\cos^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{56}{65}\right)$

Solution:

Let $\cos^{-1}\frac{12}{13} = \theta$	Let $\sin^{-1}\frac{3}{5} = \phi$
So $\cos\theta = \frac{12}{13}$	So $\sin\phi = \frac{3}{5}$
$\sin\theta = \sqrt{1-\cos^2\theta}$	$\cos\phi = \sqrt{1-\sin^2\phi}$
$= \sqrt{1-\frac{144}{169}}$	$= \sqrt{1-\frac{9}{25}}$
$= \frac{5}{13}$	$= \frac{4}{5}$

Solve the expression, Using identity: sin (θ + ϕ) = sin θ cos ϕ + cos θ sin ϕ

= 5/13 x 4/5 + 12/13 x 3/5

= (20+36)/65

= 56/65

or sin (θ + ϕ) = 56/65

or θ + ϕ) = $\sin^{-1}$ 56/65

Putting back the value of θ and ϕ, we get

$$\cos^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{56}{65}\right)$$

Hence Proved.

7. Prove that $\tan^{-1}\left(\frac{63}{16}\right) = \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right)$

Solution:

Let $\sin^{-1}\frac{5}{13} = \theta$

so $\sin\theta = \frac{5}{13}$

$\cos\theta = \sqrt{1-\sin^2\theta}$

$= \sqrt{1-\frac{25}{169}}$

$= \frac{12}{13}$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{5}{12}$

Let $\cos^{-1}\frac{3}{5} = \phi$

so $\cos\phi = \frac{3}{5}$

$\sin\phi = \sqrt{1-\cos^2\phi}$

$= \sqrt{1-\frac{9}{25}}$

$= \frac{4}{5}$

$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{4}{3}$

Solve the expression, Using identity:

$$\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi}$$

$$= \frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}\times\frac{4}{3}}$$

= 63/16

(θ + ϕ) = $\tan^{-1}$ (63/16)

Putting back the value of θ and ϕ, we get

$$\tan^{-1}\left(\frac{63}{16}\right) = \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right)$$

Hence Proved.

8. Prove that $\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$

Solution:

LHS = $\left(\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right)\right) + \left(\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{8}\right)\right)$

Solve above expressions, using below identity:

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$$

$$= \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right) + \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right)$$

After simplifying, we have

= $\tan^{-1}(6/17) + \tan^{-1}(11/23)$

Again, applying the formula, we get

$$= \tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right)$$

After simplifying,

= $\tan^{-1}(325/325)$

= $\tan^{-1}(1)$

= π/4

9. Prove that $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\frac{1-x}{1+x}$, x ∈ (0, 1)

Solution:

Let $\tan^{-1}\sqrt{x}$ = θ , then $\sqrt{x}$ = tan θ

Squaring both the sides

$\tan^2\theta = x$

Now, substitute the value of x in $\frac{1}{2}\cos^{-1}\frac{1-x}{1+x}$, we get

$$= \frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

= ½ cos-1 (cos 2 θ)

= ½ (2 θ)

= θ

$= \tan^{-1}\sqrt{x}$

10. Prove that $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}$, x ∈ (0, π/4)

Solution:

We can write 1+ sin x as,

$$1+\sin x = \cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\cos\frac{x}{2}\sin\frac{x}{2} = \left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2$$

And

$$1-\sin x = \cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\cos\frac{x}{2}\sin\frac{x}{2} = \left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2$$

LHS:

$$\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$$

$$= \cot^{-1}\left[\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)+\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)-\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}\right]$$

$= \cot^{-1}\left(\frac{2\cos(\frac{x}{2})}{2\sin(\frac{x}{2})}\right)$

$= \cot^{-1}$ (cot (x/2)

= x/2

11. Prove that $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x\ ,\ -\frac{1}{\sqrt{2}} \le x \le 1$
[Hint: Put x = cos 2 θ]

Solution:

Put $x = \cos 2\theta$ so, $\theta = \frac{1}{2}\cos^{-1} x$

$$\text{LHS} = \tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$$
$$= \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}\right)$$

Divide each term by $\sqrt{2}\cos\theta$

$$= \tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right)$$

$$= \tan^{-1}\tan\left(\frac{\pi}{4}-\theta\right)$$

$$= \frac{\pi}{4}-\theta$$

$$= \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x$$

= RHS

Hence proved

12. Prove that $\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}=\frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$

Solution:

LHS = $\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}$

$$= \frac{9}{4}\left(\frac{\pi}{2}-\sin^{-1}\frac{1}{3}\right)$$

$$= \frac{9}{4}\cos^{-1}\frac{1}{3}$$

…….(1)

(Using identity: $\sin^{-1}\theta+\cos^{-1}\theta=\frac{\pi}{2}$.)

Let θ = $\cos^{-1}$ (1/3), so cos θ = 1/3

As

$$\sin\theta=\sqrt{1-\cos^2\theta}=\sqrt{1-\frac{1}{9}}=\sqrt{\frac{8}{9}}=\frac{2\sqrt{2}}{3}$$

Using equation (1), $\frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$

Which is right hand side of the expression.

Solve the following equations:

13. 2tan^{-1} (cos x) = tan^{-1} (2 cosec x)

Solution:

$$2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$$

$$\tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}\left(\frac{2}{\sin x}\right)$$

$$\frac{2\cos x}{1-\cos^2 x} = \frac{2}{\sin x}$$

$$\frac{\cos x}{\sin x} = 1$$

Cot x = 1

x = π/4

14. Solve $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x, (x>0)$

Solution:

Put x = tan θ

$$\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x$$

This implies

$$\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x$$

$$\tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)=\frac{1}{2}\tan^{-1}\tan\theta$$

$$\tan^{-1}\left(\frac{\tan\frac{\pi}{4}-\tan\theta}{\tan\frac{\pi}{4}+\tan\theta}\right)=\frac{1}{2}\theta$$

$$\tan^{-1}\tan\left(\frac{\pi}{4}-\theta\right)=\frac{\theta}{2}$$

π/4 – θ = θ /2

or 3θ /2 = π/4

θ = π/6

Therefore, x = tan θ = tan π/6 = $1/\sqrt{3}$

15. $\sin\left(\tan^{-1}x\right), |x|<1$ **is equal to**

(A) $\frac{x}{\sqrt{1-x^2}}$ (B) $\frac{1}{\sqrt{1-x^2}}$

(C) $\frac{1}{\sqrt{1+x^2}}$ (D) $\frac{x}{\sqrt{1+x^2}}$

Solution:

Option (D) is correct.

Explanation:

Let $\theta=\tan^{-1}x$ so, $x=\tan\theta$

Again, Let's say

$\sin\left(\tan^{-1}x\right)=\sin\theta$

This implies,

$\sin\left(\tan^{-1}x\right)=\frac{1}{\cos ec\theta}=\frac{1}{\sqrt{1+\cot^2\theta}}$

Put $\cot\theta=\frac{1}{\tan\theta}=\frac{1}{x}$

Which shows,

$\sin\left(\tan^{-1}x\right)=\frac{1}{\sqrt{1+\frac{1}{x^2}}}=\frac{x}{\sqrt{x^2+1}}$

16. $\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$ **then x is equal to**

(A) 0, ½ (B) 1, ½ (C) 0 (D) ½

Solution:

Option (C) is correct.

Explanation:

Put $\sin^{-1}x=\theta$ So, $x=\sin\theta$

Now,

$\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$

$$\sin^{-1}(1-x)-2\theta=\frac{\pi}{2}$$

$$\sin^{-1}(1-x)=\frac{\pi}{2}+2\theta$$

$$1-x=\sin\left(\frac{\pi}{2}+2\theta\right)$$

$$1-x=\cos 2\theta$$

$$1-x=1-2x^2$$

(As x = sin θ)

After simplifying, we get

x(2x – 1) = 0

x = 0 or 2x – 1 = 0

x = 0 or x = ½

Equation is not true for x = ½. So the answer is x = 0.

17. $\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right)$ **is equal to**

(A) π/2 **(B) π/3** **(C) π/4** **(D) -3 π/4**

Solution:

Option (C) is correct.

Explanation:

Given expression can be written as,

$$= \tan^{-1}\left[\frac{\frac{x}{y}-\left(\frac{x-y}{x+y}\right)}{1+\frac{x}{y}\left(\frac{x-y}{x+y}\right)}\right]$$

$$= \tan^{-1}\left[\frac{x(x+y)-y(x-y)}{y(x+y)+x(x-y)}\right]$$

$$= \tan^{-1}\left(\frac{x^2+xy-xy+y^2}{xy+y^2+x^2-xy}\right)$$

$$= \tan^{-1}\left(\frac{x^2+y^2}{x^2+y^2}\right)$$

$= \tan^{-1}(1)$

$= \pi/4$