Chapter 1

12079CH01

RELATIONS AND FUNCTIONS

❖ *There is no permanent place in the world for ugly mathematics ... . It may be very hard to define mathematical beauty but that is just as true of beauty of any kind, we may not know quite what we mean by a beautiful poem, but that does not prevent us from recognising one when we read it. — G. H. HARDY* ❖

1.1 Introduction

Lejeune Dirichlet (1805-1859)

Recall that the notion of relations and functions, domain, co-domain and range have been introduced in Class XI along with different types of specific real valued functions and their graphs. The concept of the term 'relation' in mathematics has been drawn from the meaning of relation in English language, according to which two objects or quantities are related if there is a recognisable connection or link between the two objects or quantities. Let A be the set of students of Class XII of a school and B be the set of students of Class XI of the same school. Then some of the examples of relations from A to B are

(i) $\{(a, b) \in A \times B$: a is brother of $b\}$,

(ii) $\{(a, b) \in A \times B$: a is sister of $b\}$,

(iii) $\{(a, b) \in A \times B$: age of a is greater than age of $b\}$,

(iv) $\{(a, b) \in A \times B$: total marks obtained by a in the final examination is less than the total marks obtained by b in the final examination$\}$,

(v) $\{(a, b) \in A \times B$: a lives in the same locality as $b\}$. However, abstracting from this, we define mathematically a relation R from A to B as an arbitrary subset of $A \times B$.

If $(a, b) \in R$, we say that a is related to b under the relation R and we write as a R b. In general, $(a, b) \in R$, we do not bother whether there is a recognisable connection or link between a and b. As seen in Class XI, functions are special kind of relations.

In this chapter, we will study different types of relations and functions, composition of functions, invertible functions and binary operations.

1.2 Types of Relations

In this section, we would like to study different types of relations. We know that a relation in a set A is a subset of A × A. Thus, the empty set φ and A × A are two extreme relations. For illustration, consider a relation R in the set A = {1, 2, 3, 4} given by R = {(a, b): $a – b = 10$}. This is the empty set, as no pair (a, b) satisfies the condition $a – b = 10$. Similarly, R′ = {(a, b) : $| a – b | \geq 0$} is the whole set A × A, as all pairs (a, b) in A × A satisfy $| a – b | \geq 0$. These two extreme examples lead us to the following definitions.

Definition 1 A relation R in a set A is called *empty relation*, if no element of A is related to any element of A, i.e., R = φ ⊂ A × A.

Definition 2 A relation R in a set A is called *universal relation*, if each element of A is related to every element of A, i.e., R = A × A.

Both the empty relation and the universal relation are some times called *trivial relations*.

Example 1 Let A be the set of all students of a boys school. Show that the relation R in A given by R = {(a, b) : a is sister of b} is the empty relation and R′ = {(a, b) : the difference between heights of a and b is less than 3 meters} is the universal relation.

Solution Since the school is boys school, no student of the school can be sister of any student of the school. Hence, R = φ, showing that R is the empty relation. It is also obvious that the difference between heights of any two students of the school has to be less than 3 meters. This shows that R′ = A × A is the universal relation.

Remark In Class XI, we have seen two ways of representing a relation, namely raster method and set builder method. However, a relation R in the set {1, 2, 3, 4} defined by R = {(a, b) : $b = a + 1$} is also expressed as a R b if and only if $b = a + 1$ by many authors. We may also use this notation, as and when convenient.

If $(a, b) \in$ R, we say that a is related to b and we denote it as a R b.

One of the most important relation, which plays a significant role in Mathematics, is an *equivalence relation*. To study equivalence relation, we first consider three types of relations, namely reflexive, symmetric and transitive.

Definition 3 A relation R in a set A is called

(i) *reflexive*, if $(a, a) \in$ R, for every $a \in$ A,

(ii) *symmetric*, if $(a_1, a_2) \in$ R implies that $(a_2, a_1) \in$ R, for all $a_1, a_2 \in$ A.

(iii) *transitive*, if $(a_1, a_2) \in$ R and $(a_2, a_3) \in$ R implies that $(a_1, a_3) \in$ R, for all $a_1, a_2, a_3 \in$ A.

Definition 4 A relation R in a set A is said to be an *equivalence relation* if R is reflexive, symmetric and transitive.

Example 2 Let T be the set of all triangles in a plane with R a relation in T given by R = $\{(T_1, T_2) : T_1$ is congruent to $T_2\}$. Show that R is an equivalence relation.

Solution R is reflexive, since every triangle is congruent to itself. Further, $(T_1, T_2) \in R \Rightarrow T_1$ is congruent to $T_2 \Rightarrow T_2$ is congruent to $T_1 \Rightarrow (T_2, T_1) \in R$. Hence, R is symmetric. Moreover, $(T_1, T_2), (T_2, T_3) \in R \Rightarrow T_1$ is congruent to T_2 and T_2 is congruent to $T_3 \Rightarrow T_1$ is congruent to $T_3 \Rightarrow (T_1, T_3) \in R$. Therefore, R is an equivalence relation.

Example 3 Let L be the set of all lines in a plane and R be the relation in L defined as R = $\{(L_1, L_2) : L_1$ is perpendicular to $L_2\}$. Show that R is symmetric but neither reflexive nor transitive.

Solution R is not reflexive, as a line L_1 can not be perpendicular to itself, i.e., $(L_1, L_1) \notin R$. R is symmetric as $(L_1, L_2) \in R$

$\Rightarrow$ L_1 is perpendicular to L_2

$\Rightarrow$ L_2 is perpendicular to L_1

$\Rightarrow$ $(L_2, L_1) \in R$.

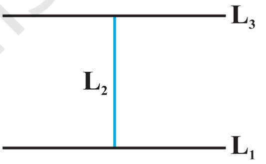


Fig 1.1

R is not transitive. Indeed, if L_1 is perpendicular to L_2 and L_2 is perpendicular to L_3, then L_1 can never be perpendicular to L_3. In fact, L_1 is parallel to L_3, i.e., $(L_1, L_2) \in R$, $(L_2, L_3) \in R$ but $(L_1, L_3) \notin R$.

Example 4 Show that the relation R in the set {1, 2, 3} given by R = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)} is reflexive but neither symmetric nor transitive.

Solution R is reflexive, since (1, 1), (2, 2) and (3, 3) lie in R. Also, R is not symmetric, as $(1, 2) \in R$ but $(2, 1) \notin R$. Similarly, R is not transitive, as $(1, 2) \in R$ and $(2, 3) \in R$ but $(1, 3) \notin R$.

Example 5 Show that the relation R in the set **Z** of integers given by

$$R = \{(a, b) : 2 \text{ divides } a - b\}$$

is an equivalence relation.

Solution R is reflexive, as 2 divides $(a - a)$ for all $a \in \mathbf{Z}$. Further, if $(a, b) \in R$, then 2 divides $a - b$. Therefore, 2 divides $b - a$. Hence, $(b, a) \in R$, which shows that R is symmetric. Similarly, if $(a, b) \in R$ and $(b, c) \in R$, then $a - b$ and $b - c$ are divisible by 2. Now, $a - c = (a - b) + (b - c)$ is even (Why?). So, $(a - c)$ is divisible by 2. This shows that R is transitive. Thus, R is an equivalence relation in **Z**.

In Example 5, note that all even integers are related to zero, as (0, ± 2), (0, ± 4) etc., lie in R and no odd integer is related to 0, as (0, ± 1), (0, ± 3) etc., do not lie in R. Similarly, all odd integers are related to one and no even integer is related to one. Therefore, the set E of all even integers and the set O of all odd integers are subsets of **Z** satisfying following conditions:

(i) All elements of E are related to each other and all elements of O are related to each other.

(ii) No element of E is related to any element of O and vice-versa.

(iii) E and O are disjoint and $\mathbf{Z} = E \cup O$.

The subset E is called the *equivalence class containing zero* and is denoted by [0]. Similarly, O is the equivalence class containing 1 and is denoted by [1]. Note that $[0] \neq [1]$, $[0] = [2r]$ and $[1] = [2r + 1]$, $r \in \mathbf{Z}$. Infact, what we have seen above is true for an arbitrary equivalence relation R in a set X. Given an arbitrary equivalence relation R in an arbitrary set X, R divides X into mutually disjoint subsets A_i called partitions or subdivisions of X satisfying:

(i) all elements of A_i are related to each other, for all i.

(ii) no element of A_i is related to any element of A_j, $i \neq j$.

(iii) $\cup A_j = X$ and $A_i \cap A_j = \phi$, $i \neq j$.

The subsets A_i are called *equivalence classes*. The interesting part of the situation is that we can go reverse also. For example, consider a subdivision of the set **Z** given by three mutually disjoint subsets A_1, A_2 and A_3 whose union is **Z** with

$$A_1 = \{x \in \mathbf{Z} : x \text{ is a multiple of } 3\} = \{..., -6, -3, 0, 3, 6, ...\}$$

$$A_2 = \{x \in \mathbf{Z} : x - 1 \text{ is a multiple of } 3\} = \{..., -5, -2, 1, 4, 7, ...\}$$

$$A_3 = \{x \in \mathbf{Z} : x - 2 \text{ is a multiple of } 3\} = \{..., -4, -1, 2, 5, 8, ...\}$$

Define a relation R in **Z** given by $R = \{(a, b) : 3 \text{ divides } a - b\}$. Following the arguments similar to those used in Example 5, we can show that R is an equivalence relation. Also, A_1 coincides with the set of all integers in **Z** which are related to zero, A_2 coincides with the set of all integers which are related to 1 and A_3 coincides with the set of all integers in **Z** which are related to 2. Thus, $A_1 = [0]$, $A_2 = [1]$ and $A_3 = [2]$. In fact, $A_1 = [3r]$, $A_2 = [3r + 1]$ and $A_3 = [3r + 2]$, for all $r \in \mathbf{Z}$.

Example 6 Let R be the relation defined in the set A = {1, 2, 3, 4, 5, 6, 7} by $R = \{(a, b) : \text{both } a \text{ and } b \text{ are either odd or even}\}$. Show that R is an equivalence relation. Further, show that all the elements of the subset {1, 3, 5, 7} are related to each other and all the elements of the subset {2, 4, 6} are related to each other, but no element of the subset {1, 3, 5, 7} is related to any element of the subset {2, 4, 6}.

Solution Given any element a in A, both a and a must be either odd or even, so that $(a, a) \in$ R. Further, $(a, b) \in$ R $\Rightarrow$ both a and b must be either odd or even $\Rightarrow (b, a) \in$ R. Similarly, $(a, b) \in$ R and $(b, c) \in$ R $\Rightarrow$ all elements a, b, c, must be either even or odd simultaneously $\Rightarrow (a, c) \in$ R. Hence, R is an equivalence relation. Further, all the elements of $\{1, 3, 5, 7\}$ are related to each other, as all the elements of this subset are odd. Similarly, all the elements of the subset $\{2, 4, 6\}$ are related to each other, as all of them are even. Also, no element of the subset $\{1, 3, 5, 7\}$ can be related to any element of $\{2, 4, 6\}$, as elements of $\{1, 3, 5, 7\}$ are odd, while elements of $\{2, 4, 6\}$ are even.

EXERCISE 1.1

1. Determine whether each of the following relations are reflexive, symmetric and transitive:
   (i) Relation R in the set A = $\{1, 2, 3, ..., 13, 14\}$ defined as
   $$R = \{(x, y) : 3x - y = 0\}$$
   (ii) Relation R in the set **N** of natural numbers defined as
   $$R = \{(x, y) : y = x + 5 \text{ and } x < 4\}$$
   (iii) Relation R in the set A = $\{1, 2, 3, 4, 5, 6\}$ as
   $$R = \{(x, y) : y \text{ is divisible by } x\}$$
   (iv) Relation R in the set **Z** of all integers defined as
   $$R = \{(x, y) : x - y \text{ is an integer}\}$$
   (v) Relation R in the set A of human beings in a town at a particular time given by
   (a) R = $\{(x, y) : x$ and y work at the same place$\}$
   (b) R = $\{(x, y) : x$ and y live in the same locality$\}$
   (c) R = $\{(x, y) : x$ is exactly 7 cm taller than $y\}$
   (d) R = $\{(x, y) : x$ is wife of $y\}$
   (e) R = $\{(x, y) : x$ is father of $y\}$
2. Show that the relation R in the set **R** of real numbers, defined as R = $\{(a, b) : a \leq b^2\}$ is neither reflexive nor symmetric nor transitive.
3. Check whether the relation R defined in the set $\{1, 2, 3, 4, 5, 6\}$ as R = $\{(a, b) : b = a + 1\}$ is reflexive, symmetric or transitive.
4. Show that the relation R in **R** defined as R = $\{(a, b) : a \leq b\}$, is reflexive and transitive but not symmetric.
5. Check whether the relation R in **R** defined by R = $\{(a, b) : a \leq b^3\}$ is reflexive, symmetric or transitive.

6. Show that the relation R in the set {1, 2, 3} given by R = {(1, 2), (2, 1)} is symmetric but neither reflexive nor transitive.

7. Show that the relation R in the set A of all the books in a library of a college, given by R = {(x, y) : x and y have same number of pages} is an equivalence relation.

8. Show that the relation R in the set A = {1, 2, 3, 4, 5} given by

R = {(a, b) : $|a - b|$ is even}, is an equivalence relation. Show that all the elements of {1, 3, 5} are related to each other and all the elements of {2, 4} are related to each other. But no element of {1, 3, 5} is related to any element of {2, 4}.

9. Show that each of the relation R in the set A = $\{x \in \mathbf{Z} : 0 \leq x \leq 12\}$, given by

(i) R = {(a, b) : $|a - b|$ is a multiple of 4}

(ii) R = {(a, b) : $a = b$}

is an equivalence relation. Find the set of all elements related to 1 in each case.

10. Give an example of a relation. Which is

(i) Symmetric but neither reflexive nor transitive.

(ii) Transitive but neither reflexive nor symmetric.

(iii) Reflexive and symmetric but not transitive.

(iv) Reflexive and transitive but not symmetric.

(v) Symmetric and transitive but not reflexive.

11. Show that the relation R in the set A of points in a plane given by R = {(P, Q) : distance of the point P from the origin is same as the distance of the point Q from the origin}, is an equivalence relation. Further, show that the set of all points related to a point P $\neq$ (0, 0) is the circle passing through P with origin as centre.

12. Show that the relation R defined in the set A of all triangles as R = {(T_1, T_2) : T_1 is similar to T_2}, is equivalence relation. Consider three right angle triangles T_1 with sides 3, 4, 5, T_2 with sides 5, 12, 13 and T_3 with sides 6, 8, 10. Which triangles among T_1, T_2 and T_3 are related?

13. Show that the relation R defined in the set A of all polygons as R = {(P_1, P_2) : P_1 and P_2 have same number of sides}, is an equivalence relation. What is the set of all elements in A related to the right angle triangle T with sides 3, 4 and 5?

14. Let L be the set of all lines in XY plane and R be the relation in L defined as R = {(L_1, L_2) : L_1 is parallel to L_2}. Show that R is an equivalence relation. Find the set of all lines related to the line $y = 2x + 4$.

15. Let R be the relation in the set {1, 2, 3, 4} given by R = {(1, 2), (2, 2), (1, 1), (4,4), (1, 3), (3, 3), (3, 2)}. Choose the correct answer.

(A) R is reflexive and symmetric but not transitive.

(B) R is reflexive and transitive but not symmetric.

(C) R is symmetric and transitive but not reflexive.

(D) R is an equivalence relation.

16. Let R be the relation in the set **N** given by $R = \{(a, b) : a = b - 2, b > 6\}$. Choose the correct answer.

(A) $(2, 4) \in R$ (B) $(3, 8) \in R$ (C) $(6, 8) \in R$ (D) $(8, 7) \in R$

1.3 Types of Functions

The notion of a function along with some special functions like identity function, constant function, polynomial function, rational function, modulus function, signum function etc. along with their graphs have been given in Class XI.

Addition, subtraction, multiplication and division of two functions have also been studied. As the concept of function is of paramount importance in mathematics and among other disciplines as well, we would like to extend our study about function from where we finished earlier. In this section, we would like to study different types of functions.

Consider the functions f_1, f_2, f_3 and f_4 given by the following diagrams.

In Fig 1.2, we observe that the images of distinct elements of X_1 under the function f_1 are distinct, but the image of two distinct elements 1 and 2 of X_1 under f_2 is same, namely b. Further, there are some elements like e and f in X_2 which are not images of any element of X_1 under f_1, while all elements of X_3 are images of some elements of X_1 under f_3. The above observations lead to the following definitions:

Definition 5 A function $f : X \to Y$ is defined to be *one-one* (or *injective*), if the images of distinct elements of X under f are distinct, i.e., for every $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Otherwise, f is called *many-one*.

The function f_1 and f_4 in Fig 1.2 (i) and (iv) are one-one and the function f_2 and f_3 in Fig 1.2 (ii) and (iii) are many-one.

Definition 6 A function $f : X \to Y$ is said to be *onto* (or *surjective*), if every element of Y is the image of some element of X under f, i.e., for every $y \in Y$, there exists an element x in X such that $f(x) = y$.

The function f_3 and f_4 in Fig 1.2 (iii), (iv) are onto and the function f_1 in Fig 1.2 (i) is not onto as elements e, f in X_2 are not the image of any element in X_1 under f_1.

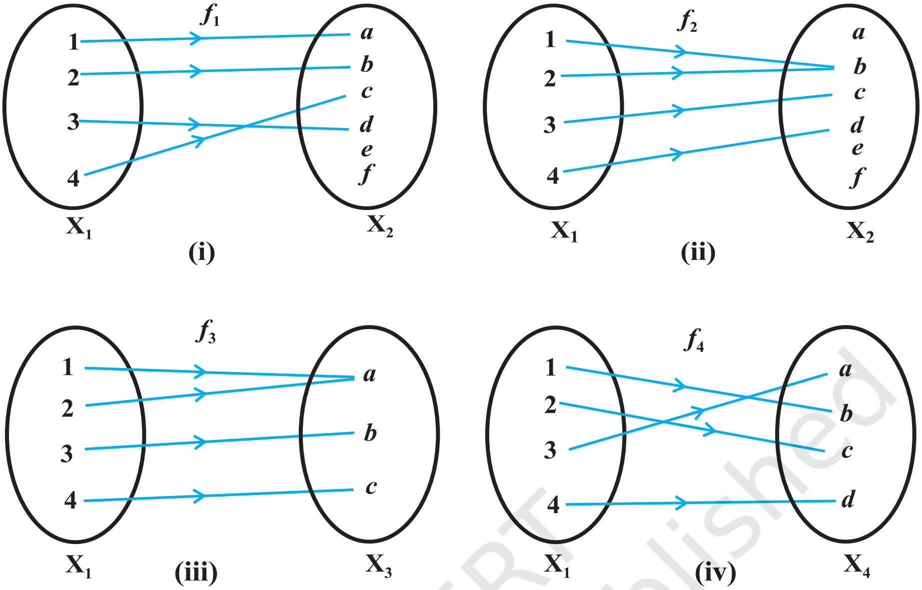


Fig 1.2 (i) to (iv)

Remark $f : X \rightarrow Y$ is onto if and only if Range of $f = Y$.

Definition 7 A function $f : X \rightarrow Y$ is said to be *one-one* and *onto* (or *bijective*), if f is both one-one and onto.

The function f_4 in Fig 1.2 (iv) is one-one and onto.

Example 7 Let A be the set of all 50 students of Class X in a school. Let $f : A \rightarrow \mathbf{N}$ be function defined by $f(x)$ = roll number of the student x. Show that f is one-one but not onto.

Solution No two different students of the class can have same roll number. Therefore, f must be one-one. We can assume without any loss of generality that roll numbers of students are from 1 to 50. This implies that 51 in **N** is not roll number of any student of the class, so that 51 can not be image of any element of X under f. Hence, f is not onto.

Example 8 Show that the function $f : \mathbf{N} \rightarrow \mathbf{N}$, given by $f(x) = 2x$, is one-one but not onto.

Solution The function f is one-one, for $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$. Further, f is not onto, as for $1 \in \mathbf{N}$, there does not exist any x in **N** such that $f(x) = 2x = 1$.

Example 9 Prove that the function $f : \mathbf{R} \to \mathbf{R}$, given by $f(x) = 2x$, is one-one and onto.

Solution f is one-one, as $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$. Also, given any real number y in R, there exists $\frac{y}{2}$ in R such that $f(\frac{y}{2}) = 2 \cdot (\frac{y}{2}) = y$. Hence, f is onto.

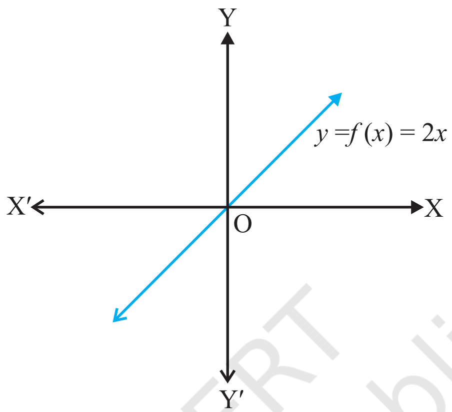


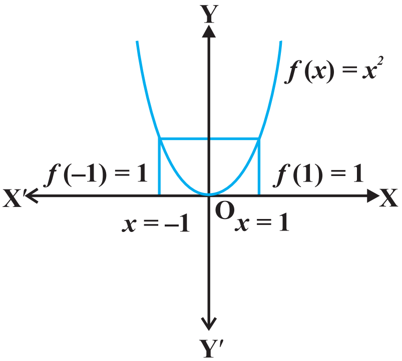
Fig 1.3

Example 10 Show that the function $f : \mathbf{N} \to \mathbf{N}$, given by $f(1) = f(2) = 1$ and $f(x) = x - 1$, for every $x > 2$, is onto but not one-one.

Solution f is not one-one, as $f(1) = f(2) = 1$. But f is onto, as given any $y \in \mathbf{N}$, $y \neq 1$, we can choose x as $y + 1$ such that $f(y + 1) = y + 1 - 1 = y$. Also for $1 \in \mathbf{N}$, we have $f(1) = 1$.

Example 11 Show that the function $f : \mathbf{R} \to \mathbf{R}$, defined as $f(x) = x^2$, is neither one-one nor onto.

Solution Since $f(-1) = 1 = f(1)$, f is not one-one. Also, the element -2 in the co-domain **R** is not image of any element x in the domain **R** (Why?). Therefore f is not onto.



The image of 1 and –1 under f is 1.

Fig 1.4

Example 12 Show that $f : \mathbf{N} \to \mathbf{N}$, given by

$$f(x) = \begin{cases} x + 1, \text{ if } x \text{ is odd,} \\ x - 1, \text{ if } x \text{ is even} \end{cases}$$

is both one-one and onto.

Solution Suppose $f(x_1) = f(x_2)$. Note that if x_1 is odd and x_2 is even, then we will have $x_1 + 1 = x_2 - 1$, i.e., $x_2 - x_1 = 2$ which is impossible. Similarly, the possibility of x_1 being even and x_2 being odd can also be ruled out, using the similar argument. Therefore, both x_1 and x_2 must be either odd or even. Suppose both x_1 and x_2 are odd. Then $f(x_1) = f(x_2) \Rightarrow x_1 + 1 = x_2 + 1 \Rightarrow x_1 = x_2$. Similarly, if both x_1 and x_2 are even, then also $f(x_1) = f(x_2) \Rightarrow x_1 - 1 = x_2 - 1 \Rightarrow x_1 = x_2$. Thus, f is one-one. Also, any odd number $2r + 1$ in the co-domain **N** is the image of $2r + 2$ in the domain **N** and any even number $2r$ in the co-domain **N** is the image of $2r - 1$ in the domain **N**. Thus, f is onto.

Example 13 Show that an onto function $f: \{1, 2, 3\} \to \{1, 2, 3\}$ is always one-one.

Solution Suppose f is not one-one. Then there exists two elements, say 1 and 2 in the domain whose image in the co-domain is same. Also, the image of 3 under f can be only one element. Therefore, the range set can have at the most two elements of the co-domain $\{1, 2, 3\}$, showing that f is not onto, a contradiction. Hence, f must be one-one.

Example 14 Show that a one-one function $f: \{1, 2, 3\} \to \{1, 2, 3\}$ must be onto.

Solution Since f is one-one, three elements of $\{1, 2, 3\}$ must be taken to 3 different elements of the co-domain $\{1, 2, 3\}$ under f. Hence, f has to be onto.

Remark The results mentioned in Examples 13 and 14 are also true for an arbitrary finite set X, i.e., a one-one function $f: X \to X$ is necessarily onto and an onto map $f: X \to X$ is necessarily one-one, for every finite set X. In contrast to this, Examples 8 and 10 show that for an infinite set, this may not be true. In fact, this is a characteristic difference between a finite and an infinite set.

EXERCISE 1.2

1. Show that the function $f: \mathbf{R}_* \to \mathbf{R}_*$ defined by $f(x) = \dfrac{1}{x}$ is one-one and onto, where $\mathbf{R}_*$ is the set of all non-zero real numbers. Is the result true, if the domain $\mathbf{R}_*$ is replaced by **N** with co-domain being same as $\mathbf{R}_*$?
2. Check the injectivity and surjectivity of the following functions:
   (i) $f: \mathbf{N} \to \mathbf{N}$ given by $f(x) = x^2$
   (ii) $f: \mathbf{Z} \to \mathbf{Z}$ given by $f(x) = x^2$
   (iii) $f: \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^2$
   (iv) $f: \mathbf{N} \to \mathbf{N}$ given by $f(x) = x^3$
   (v) $f: \mathbf{Z} \to \mathbf{Z}$ given by $f(x) = x^3$
3. Prove that the Greatest Integer Function $f: \mathbf{R} \to \mathbf{R}$, given by $f(x) = [x]$, is neither one-one nor onto, where $[x]$ denotes the greatest integer less than or equal to x.

4. Show that the Modulus Function $f : \mathbf{R} \to \mathbf{R}$, given by $f(x) = |x|$, is neither one-one nor onto, where $|x|$ is x, if x is positive or 0 and $|x|$ is $-x$, if x is negative.

5. Show that the Signum Function $f : \mathbf{R} \to \mathbf{R}$, given by

$$f(x) = \begin{cases} 1, \text{ if } x > 0 \\ 0, \text{ if } x = 0 \\ 1, \text{ if } x < 0 \end{cases}$$

is neither one-one nor onto.

6. Let A = {1, 2, 3}, B = {4, 5, 6, 7} and let f = {(1, 4), (2, 5), (3, 6)} be a function from A to B. Show that f is one-one.

7. In each of the following cases, state whether the function is one-one, onto or bijective. Justify your answer.

(i) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 3 - 4x$

(ii) $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 1 + x^2$

8. Let A and B be sets. Show that $f : A \times B \to B \times A$ such that $f(a, b) = (b, a)$ is bijective function.

9. Let $f : \mathbf{N} \to \mathbf{N}$ be defined by $f(n) = \begin{cases} \dfrac{n+1}{2}, \text{ if } n \text{ is odd} \\ \dfrac{n}{2}, \text{ if } n \text{ is even} \end{cases}$ for all $n \in \mathbf{N}$.

State whether the function f is bijective. Justify your answer.

10. Let A = $\mathbf{R}$ – {3} and B = $\mathbf{R}$ – {1}. Consider the function $f : A \to B$ defined by $f(x) = \left(\dfrac{x-2}{x-3}\right)$. Is f one-one and onto? Justify your answer.

11. Let $f : \mathbf{R} \to \mathbf{R}$ be defined as $f(x) = x^4$. Choose the correct answer.

(A) f is one-one onto
(B) f is many-one onto
(C) f is one-one but not onto
(D) f is neither one-one nor onto.

12. Let $f : \mathbf{R} \to \mathbf{R}$ be defined as $f(x) = 3x$. Choose the correct answer.

(A) f is one-one onto
(B) f is many-one onto
(C) f is one-one but not onto
(D) f is neither one-one nor onto.

1.4 Composition of Functions and Invertible Function

Definition 8 Let $f: A \to B$ and $g : B \to C$ be two functions. Then the composition of f and g, denoted by gof, is defined as the function $gof : A \to C$ given by

$$gof(x) = g(f(x)), \ \forall \ x \in A.$$

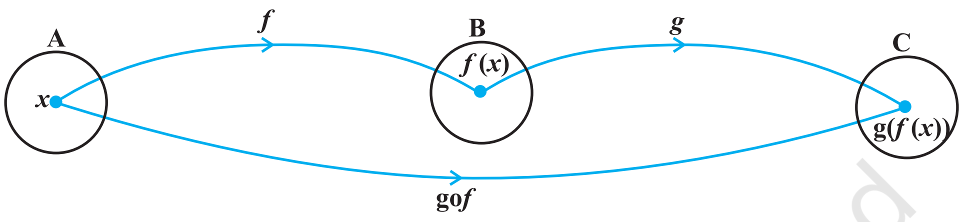


Fig 1.5

Example 15 Let $f: \{2, 3, 4, 5\} \to \{3, 4, 5, 9\}$ and $g : \{3, 4, 5, 9\} \to \{7, 11, 15\}$ be functions defined as $f(2) = 3$, $f(3) = 4$, $f(4) = f(5) = 5$ and $g(3) = g(4) = 7$ and $g(5) = g(9) = 11$. Find gof.

Solution We have $gof(2) = g(f(2)) = g(3) = 7$, $gof(3) = g(f(3)) = g(4) = 7$, $gof(4) = g(f(4)) = g(5) = 11$ and $gof(5) = g(5) = 11$.

Example 16 Find gof and fog, if $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ are given by $f(x) = \cos x$ and $g(x) = 3x^2$. Show that $gof \neq fog$.

Solution We have $gof(x) = g(f(x)) = g(\cos x) = 3 (\cos x)^2 = 3 \cos^2 x$. Similarly, $fog(x) = f(g(x)) = f(3x^2) = \cos (3x^2)$. Note that $3\cos^2 x \neq \cos 3x^2$, for $x = 0$. Hence, $gof \neq fog$.

Definition 9 A function $f : X \to Y$ is defined to be *invertible*, if there exists a function $g : Y \to X$ such that $gof = I_X$ and $fog = I_Y$. The function g is called the *inverse of f* and is denoted by f^{-1}.

Thus, if f is invertible, then f must be one-one and onto and conversely, if f is one-one and onto, then f must be invertible. This fact significantly helps for proving a function f to be invertible by showing that f is one-one and onto, specially when the actual inverse of f is not to be determined.

Example 17 Let $f : \mathbf{N} \to Y$ be a function defined as $f(x) = 4x + 3$, where, $Y = \{y \in \mathbf{N} : y = 4x + 3 \text{ for some } x \in \mathbf{N}\}$. Show that f is invertible. Find the inverse.

Solution Consider an arbitrary element y of Y. By the definition of Y, $y = 4x + 3$, for some x in the domain $\mathbf{N}$. This shows that $x = \frac{(y-3)}{4}$. Define $g : Y \to \mathbf{N}$ by

$g(y) = \frac{(y-3)}{4}$. Now, $gof(x) = g(f(x)) = g(4x + 3) = \frac{(4x+3-3)}{4} = x$ and

$fog(y) = f(g(y)) = f\left(\frac{(y-3)}{4}\right) = \frac{4(y-3)}{4} + 3 = y - 3 + 3 = y$. This shows that $gof = I_N$ and $fog = I_Y$, which implies that f is invertible and g is the inverse of f.

Miscellaneous Examples

Example 18 If R_1 and R_2 are equivalence relations in a set A, show that $R_1 \cap R_2$ is also an equivalence relation.

Solution Since R_1 and R_2 are equivalence relations, $(a, a) \in R_1$, and $(a, a) \in R_2$ $\forall a \in A$. This implies that $(a, a) \in R_1 \cap R_2$, $\forall a$, showing $R_1 \cap R_2$ is reflexive. Further, $(a, b) \in R_1 \cap R_2 \Rightarrow (a, b) \in R_1$ and $(a, b) \in R_2 \Rightarrow (b, a) \in R_1$ and $(b, a) \in R_2 \Rightarrow (b, a) \in R_1 \cap R_2$, hence, $R_1 \cap R_2$ is symmetric. Similarly, $(a, b) \in R_1 \cap R_2$ and $(b, c) \in R_1 \cap R_2 \Rightarrow (a, c) \in R_1$ and $(a, c) \in R_2 \Rightarrow (a, c) \in R_1 \cap R_2$. This shows that $R_1 \cap R_2$ is transitive. Thus, $R_1 \cap R_2$ is an equivalence relation.

Example 19 Let R be a relation on the set A of ordered pairs of positive integers defined by (x, y) R (u, v) if and only if $xv = yu$. Show that R is an equivalence relation.

Solution Clearly, (x, y) R (x, y), $\forall (x, y) \in A$, since $xy = yx$. This shows that R is reflexive. Further, (x, y) R $(u, v) \Rightarrow xv = yu \Rightarrow uy = vx$ and hence (u, v) R (x, y). This shows that R is symmetric. Similarly, (x, y) R (u, v) and (u, v) R $(a, b) \Rightarrow xv = yu$ and $ub = va \Rightarrow xv\frac{a}{u} = yu\frac{a}{u} \Rightarrow xv\frac{b}{v} = yu\frac{a}{u} \Rightarrow xb = ya$ and hence (x, y) R (a, b). Thus, R is transitive. Thus, R is an equivalence relation.

Example 20 Let X = {1, 2, 3, 4, 5, 6, 7, 8, 9}. Let R_1 be a relation in X given by $R_1 = \{(x, y) : x - y$ is divisible by 3$\}$ and R_2 be another relation on X given by $R_2 = \{(x, y): \{x, y\} \subset \{1, 4, 7\}\}$ or $\{x, y\} \subset \{2, 5, 8\}$ or $\{x, y\} \subset \{3, 6, 9\}\}$. Show that $R_1 = R_2$.

Solution Note that the characteristic of sets {1, 4, 7}, {2, 5, 8} and {3, 6, 9} is that difference between any two elements of these sets is a multiple of 3. Therefore, $(x, y) \in R_1 \Rightarrow x - y$ is a multiple of 3 $\Rightarrow \{x, y\} \subset \{1, 4, 7\}$ or $\{x, y\} \subset \{2, 5, 8\}$ or $\{x, y\} \subset \{3, 6, 9\} \Rightarrow (x, y) \in R_2$. Hence, $R_1 \subset R_2$. Similarly, $\{x, y\} \in R_2 \Rightarrow \{x, y\}$

$\subset \{1, 4, 7\}$ or $\{x, y\} \subset \{2, 5, 8\}$ or $\{x, y\} \subset \{3, 6, 9\} \Rightarrow x - y$ is divisible by $3 \Rightarrow \{x, y\} \in R_1$. This shows that $R_2 \subset R_1$. Hence, $R_1 = R_2$.

Example 21 Let $f: X \to Y$ be a function. Define a relation R in X given by $R = \{(a, b): f(a) = f(b)\}$. Examine whether R is an equivalence relation or not.

Solution For every $a \in X$, $(a, a) \in R$, since $f(a) = f(a)$, showing that R is reflexive. Similarly, $(a, b) \in R \Rightarrow f(a) = f(b) \Rightarrow f(b) = f(a) \Rightarrow (b, a) \in R$. Therefore, R is symmetric. Further, $(a, b) \in R$ and $(b, c) \in R \Rightarrow f(a) = f(b)$ and $f(b) = f(c) \Rightarrow f(a) = f(c) \Rightarrow (a, c) \in R$, which implies that R is transitive. Hence, R is an equivalence relation.

Example 22 Find the number of all one-one functions from set $A = \{1, 2, 3\}$ to itself.

Solution One-one function from $\{1, 2, 3\}$ to itself is simply a permutation on three symbols 1, 2, 3. Therefore, total number of one-one maps from $\{1, 2, 3\}$ to itself is same as total number of permutations on three symbols 1, 2, 3 which is $3! = 6$.

Example 23 Let $A = \{1, 2, 3\}$. Then show that the number of relations containing (1, 2) and (2, 3) which are reflexive and transitive but not symmetric is three.

Solution The smallest relation R_1 containing (1, 2) and (2, 3) which is reflexive and transitive but not symmetric is $\{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$. Now, if we add the pair (2, 1) to R_1 to get R_2, then the relation R_2 will be reflexive, transitive but not symmetric. Similarly, we can obtain R_3 by adding (3, 2) to R_1 to get the desired relation. However, we can not add two pairs (2, 1), (3, 2) or single pair (3, 1) to R_1 at a time, as by doing so, we will be forced to add the remaining pair in order to maintain transitivity and in the process, the relation will become symmetric also which is not required. Thus, the total number of desired relations is three.

Example 24 Show that the number of equivalence relation in the set $\{1, 2, 3\}$ containing (1, 2) and (2, 1) is two.

Solution The smallest equivalence relation R_1 containing (1, 2) and (2, 1) is $\{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$. Now we are left with only 4 pairs namely (2, 3), (3, 2), (1, 3) and (3, 1). If we add any one, say (2, 3) to R_1, then for symmetry we must add (3, 2) also and now for transitivity we are forced to add (1, 3) and (3, 1). Thus, the only equivalence relation bigger than R_1 is the universal relation. This shows that the total number of equivalence relations containing (1, 2) and (2, 1) is two.

Example 25 Consider the identity function $I_{\mathbf{N}} : \mathbf{N} \to \mathbf{N}$ defined as $I_{\mathbf{N}}(x) = x \ \forall x \in \mathbf{N}$. Show that although $I_{\mathbf{N}}$ is onto but $I_{\mathbf{N}} + I_{\mathbf{N}} : \mathbf{N} \to \mathbf{N}$ defined as

$$(I_{\mathbf{N}} + I_{\mathbf{N}})(x) = I_{\mathbf{N}}(x) + I_{\mathbf{N}}(x) = x + x = 2x \text{ is not onto.}$$

Solution Clearly I_N is onto. But $I_N + I_N$ is not onto, as we can find an element 3 in the co-domain **N** such that there does not exist any x in the domain **N** with $(I_N + I_N)(x) = 2x = 3$.

Example 26 Consider a function $f : \left[0, \frac{\pi}{2}\right] \to \mathbf{R}$ given by $f(x) = \sin x$ and $g : \left[0, \frac{\pi}{2}\right] \to \mathbf{R}$ given by $g(x) = \cos x$. Show that f and g are one-one, but $f + g$ is not one-one.

Solution Since for any two distinct elements x_1 and x_2 in $\left[0, \frac{\pi}{2}\right]$, $\sin x_1 \neq \sin x_2$ and $\cos x_1 \neq \cos x_2$, both f and g must be one-one. But $(f + g)(0) = \sin 0 + \cos 0 = 1$ and $(f+g)\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \cos\frac{\pi}{2} = 1$. Therefore, $f + g$ is not one-one.

Miscellaneous Exercise on Chapter 1

1. Show that the function $f : \mathbf{R} \to \{x \in \mathbf{R} : -1 < x < 1\}$ defined by $f(x) = \frac{x}{1 + |x|}$, $x \in \mathbf{R}$ is one one and onto function.

2. Show that the function $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^3$ is injective.

3. Given a non empty set X, consider P(X) which is the set of all subsets of X. Define the relation R in P(X) as follows:

For subsets A, B in P(X), ARB if and only if $A \subset B$. Is R an equivalence relation on P(X)? Justify your answer.

4. Find the number of all onto functions from the set $\{1, 2, 3, ..., n\}$ to itself.

5. Let $A = \{-1, 0, 1, 2\}$, $B = \{-4, -2, 0, 2\}$ and $f, g : A \to B$ be functions defined by $f(x) = x^2 - x$, $x \in A$ and $g(x) = 2\left|x - \frac{1}{2}\right| - 1$, $x \in A$. Are f and g equal? Justify your answer. (Hint: One may note that two functions $f : A \to B$ and $g : A \to B$ such that $f(a) = g(a)\ \forall\ a \in A$, are called equal functions).

6. Let A = {1, 2, 3}. Then number of relations containing (1, 2) and (1, 3) which are reflexive and symmetric but not transitive is

(A) 1 (B) 2 (C) 3 (D) 4

7. Let A = {1, 2, 3}. Then number of equivalence relations containing (1, 2) is

(A) 1 (B) 2 (C) 3 (D) 4

Summary

In this chapter, we studied different types of relations and equivalence relation, composition of functions, invertible functions and binary operations. The main features of this chapter are as follows:

- *Empty relation* is the relation R in X given by $R = \phi \subset X \times X$.
- *Universal relation* is the relation R in X given by $R = X \times X$.
- *Reflexive relation* R in X is a relation with $(a, a) \in R \ \forall \ a \in X$.
- *Symmetric relation* R in X is a relation satisfying $(a, b) \in R$ implies $(b, a) \in R$.
- *Transitive relation* R in X is a relation satisfying $(a, b) \in R$ and $(b, c) \in R$ implies that $(a, c) \in R$.
- *Equivalence relation* R in X is a relation which is reflexive, symmetric and transitive.
- *Equivalence class* $[a]$ containing $a \in X$ for an equivalence relation R in X is the subset of X containing all elements b related to a.
- A function $f : X \to Y$ is *one-one* (or *injective*) if
  $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \ \forall \ x_1, x_2 \in X$.
- A function $f : X \to Y$ is *onto* (or *surjective*) if given any $y \in Y$, $\exists \ x \in X$ such that $f(x) = y$.
- A function $f : X \to Y$ is *one-one and onto* (or *bijective*), if f is both one-one and onto.
- Given a finite set X, a function $f : X \to X$ is one-one (respectively onto) if and only if f is onto (respectively one-one). This is the characteristic property of a finite set. This is not true for infinite set

Historical Note

The concept of function has evolved over a long period of time starting from R. Descartes (1596-1650), who used the word 'function' in his manuscript "*Geometrie*" in 1637 to mean some positive integral power x^n of a variable x while studying geometrical curves like hyperbola, parabola and ellipse. James Gregory (1636-1675) in his work " *Vera Circuli et Hyperbolae Quadratura*" (1667) considered function as a quantity obtained from other quantities by successive use of algebraic operations or by any other operations. Later G. W. Leibnitz (1646-1716) in his manuscript "*Methodus tangentium inversa, seu de functionibus*" written in 1673 used the word 'function' to mean a quantity varying from point to point on a curve such as the coordinates of a point on the curve, the slope of the curve, the tangent and the normal to the curve at a point. However, in his manuscript "*Historia*" (1714), Leibnitz used the word 'function' to mean quantities that depend on a variable. He was the first to use the phrase 'function of x'. John Bernoulli (1667-1748) used the notation ϕx for the first time in 1718 to indicate a function of x. But the general adoption of symbols like f, F, ϕ, ψ ... to represent functions was made by Leonhard Euler (1707-1783) in 1734 in the first part of his manuscript "*Analysis Infinitorium*". Later on, Joeph Louis Lagrange (1736-1813) published his manuscripts "*Theorie des functions analytiques*" in 1793, where he discussed about analytic function and used the notion $f(x)$, F(x), $\phi(x)$ etc. for different function of x. Subsequently, Lejeunne Dirichlet (1805-1859) gave the definition of function which was being used till the set theoretic definition of function presently used, was given after set theory was developed by Georg Cantor (1845-1918). The set theoretic definition of function known to us presently is simply an abstraction of the definition given by Dirichlet in a rigorous manner.

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Chapter 2

INVERSE TRIGONOMETRIC FUNCTIONS

❖ *Mathematics, in general, is fundamentally the science of self-evident things. — FELIX KLEIN* ❖

2.1 Introduction


Aryabhata (476-550 A. D.)

In Chapter 1, we have studied that the inverse of a function f, denoted by f^{-1}, exists if f is one-one and onto. There are many functions which are not one-one, onto or both and hence we can not talk of their inverses. In Class XI, we studied that trigonometric functions are not one-one and onto over their natural domains and ranges and hence their inverses do not exist. In this chapter, we shall study about the restrictions on domains and ranges of trigonometric functions which ensure the existence of their inverses and observe their behaviour through graphical representations. Besides, some elementary properties will also be discussed.

The inverse trigonometric functions play an important role in calculus for they serve to define many integrals. The concepts of inverse trigonometric functions is also used in science and engineering.

2.2 Basic Concepts

In Class XI, we have studied trigonometric functions, which are defined as follows:

sine function, i.e., sine : $\mathbf{R} \to [-1, 1]$

cosine function, i.e., cos : $\mathbf{R} \to [-1, 1]$

tangent function, i.e., tan : $\mathbf{R} - \{x : x = (2n+1)\frac{\pi}{2}, n \in \mathbf{Z}\} \to \mathbf{R}$

cotangent function, i.e., cot : $\mathbf{R} - \{x : x = n\pi, n \in \mathbf{Z}\} \to \mathbf{R}$

secant function, i.e., sec : $\mathbf{R} - \{x : x = (2n+1)\frac{\pi}{2}, n \in \mathbf{Z}\} \to \mathbf{R} - (-1, 1)$

cosecant function, i.e., cosec : $\mathbf{R} - \{x : x = n\pi, n \in \mathbf{Z}\} \to \mathbf{R} - (-1, 1)$

We have also learnt in Chapter 1 that if $f : X \to Y$ such that $f(x) = y$ is one-one and onto, then we can define a unique function $g : Y \to X$ such that $g(y) = x$, where $x \in X$ and $y = f(x)$, $y \in Y$. Here, the domain of g = range of f and the range of g = domain of f. The function g is called the inverse of f and is denoted by f^{-1}. Further, g is also one-one and onto and inverse of g is f. Thus, $g^{-1} = (f^{-1})^{-1} = f$. We also have

$$(f^{-1} \text{ o } f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x$$

and $$(f \text{ o } f^{-1})(y) = f(f^{-1}(y)) = f(x) = y$$

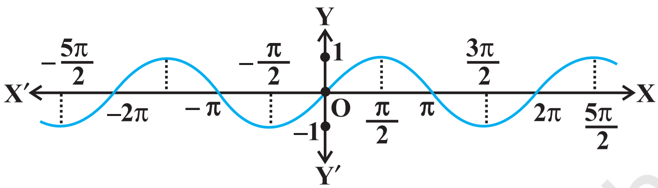
Since the domain of sine function is the set of all real numbers and range is the closed interval [–1, 1]. If we restrict its domain to $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, then it becomes one-one and onto with range [– 1, 1]. Actually, sine function restricted to any of the intervals $\left[\frac{-3\pi}{2}, \frac{\pi}{2}\right]$, $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ etc., is one-one and its range is [–1, 1]. We can, therefore, define the inverse of sine function in each of these intervals. We denote the inverse of sine function by $\sin^{-1}$ (arc sine function). Thus, $\sin^{-1}$ is a function whose domain is [– 1, 1] and range could be any of the intervals $\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right]$, $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ or $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, and so on. Corresponding to each such interval, we get a *branch* of the function $\sin^{-1}$. The branch with range $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is called the *principal value branch, whereas other intervals as range give different branches of* $\sin^{-1}$. When we refer to the function $\sin^{-1}$, we take it as the function whose domain is [–1, 1] and range is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. We write $\sin^{-1} : [-1, 1] \to \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

From the definition of the inverse functions, it follows that $\sin(\sin^{-1} x) = x$ if $-1 \le x \le 1$ and $\sin^{-1}(\sin x) = x$ if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. In other words, if $y = \sin^{-1} x$, then $\sin y = x$.

Remarks

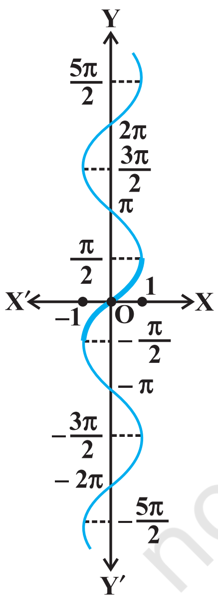
(i) We know from Chapter 1, that if $y = f(x)$ is an invertible function, then $x = f^{-1}(y)$. Thus, the graph of $\sin^{-1}$ function can be obtained from the graph of original function by interchanging x and y axes, i.e., if (a, b) is a point on the graph of sine function, then (b, a) becomes the corresponding point on the graph of inverse of sine function. Thus, the graph of the function $y = \sin^{-1} x$ can be obtained from the graph of $y = \sin x$ by interchanging x and y axes. The graphs of $y = \sin x$ and $y = \sin^{-1} x$ are as given in Fig 2.1 (i), (ii), (iii). The dark portion of the graph of $y = \sin^{-1} x$ represent the principal value branch.

(ii) It can be shown that the graph of an inverse function can be obtained from the corresponding graph of original function as a mirror image (i.e., reflection) along the line $y = x$. This can be visualised by looking the graphs of $y = \sin x$ and $y = \sin^{-1} x$ as given in the same axes (Fig 2.1 (iii)).



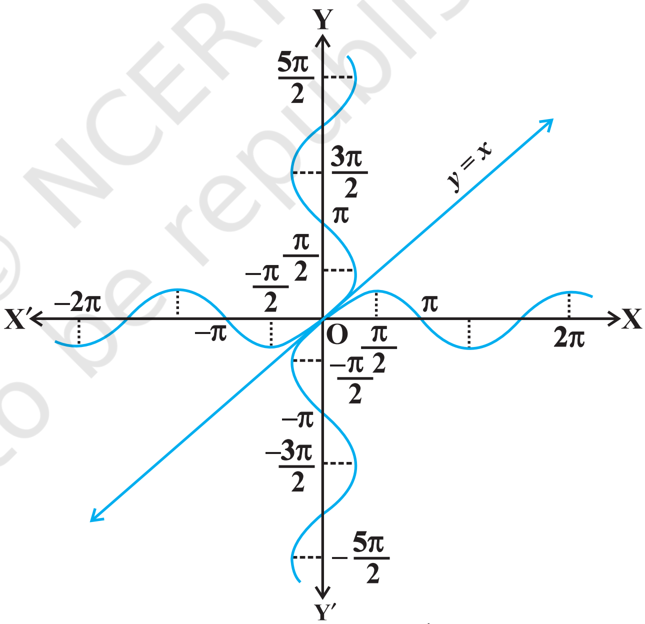
$y = \sin x$

Fig 2.1 (i)



$y = \sin^{-1} x$

Fig 2.1 (ii)



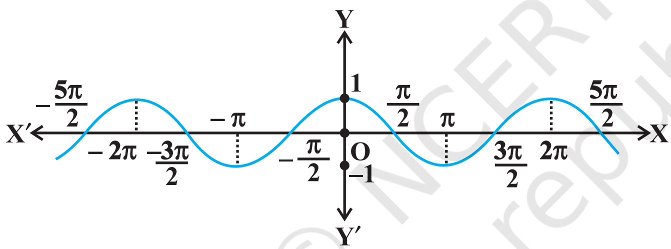
$y = \sin x$ and $y = \sin^{-1} x$

Fig 2.1 (iii)

Like sine function, the cosine function is a function whose domain is the set of all real numbers and range is the set [–1, 1]. If we restrict the domain of cosine function to [0, π], then it becomes one-one and onto with range [–1, 1]. Actually, cosine function restricted to any of the intervals $[-\pi, 0]$, $[0,\pi]$, $[\pi, 2\pi]$ etc., is bijective with range as $[-1, 1]$. We can, therefore, define the inverse of cosine function in each of these intervals. We denote the inverse of the cosine function by $\cos^{-1}$ (arc cosine function). Thus, $\cos^{-1}$ is a function whose domain is $[-1, 1]$ and range could be any of the intervals $[-\pi, 0]$, $[0, \pi]$, $[\pi, 2\pi]$ etc. Corresponding to each such interval, we get a branch of the function $\cos^{-1}$. The branch with range $[0, \pi]$ is called the *principal value branch* of the function $\cos^{-1}$. We write

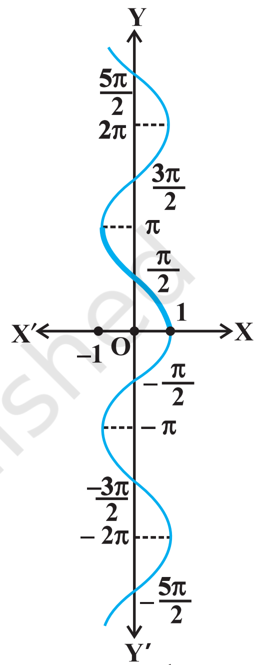
$$\cos^{-1} : [-1, 1] \to [0, \pi].$$

The graph of the function given by $y = \cos^{-1} x$ can be drawn in the same way as discussed about the graph of $y = \sin^{-1} x$. The graphs of $y = \cos x$ and $y = \cos^{-1} x$ are given in Fig 2.2 (i) and (ii).



$y = \cos x$

Fig 2.2 (i)



$y = \cos^{-1} x$

Fig 2.2 (ii)

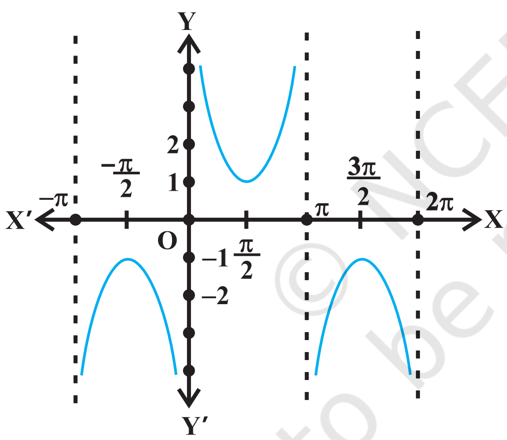
Let us now discuss $\text{cosec}^{-1}x$ and $\sec^{-1}x$ as follows:

Since, $\text{cosec}\, x = \dfrac{1}{\sin x}$, the domain of the cosec function is the set $\{x : x \in \mathbf{R}$ and $x \neq n\pi,\ n \in \mathbf{Z}\}$ and the range is the set $\{y : y \in \mathbf{R},\ y \geq 1$ or $y \leq -1\}$ i.e., the set $\mathbf{R} - (-1, 1)$. It means that $y = \text{cosec}\, x$ assumes all real values except $-1 < y < 1$ and is not defined for integral multiple of π. If we restrict the domain of cosec function to $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] - \{0\}$, then it is one to one and onto with its range as the set $\mathbf{R} - (-1, 1)$. Actually, cosec function restricted to any of the intervals $\left[\dfrac{-3\pi}{2}, \dfrac{-\pi}{2}\right] - \{-\pi\}$, $\left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right] - \{0\}$, $\left[\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right] - \{\pi\}$ etc., is bijective and its range is the set of all real numbers $\mathbf{R} - (-1, 1)$.

Thus cosec^{-1} can be defined as a function whose domain is **R** – (–1, 1) and range could be any of the intervals $\left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right] - \{-\pi\}$, $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$, $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right] - \{\pi\}$ etc. The function corresponding to the range $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ is called the *principal value branch* of cosec^{-1}. We thus have principal branch as

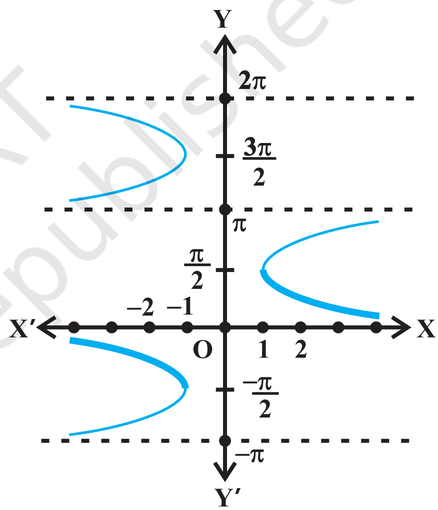
$$\text{cosec}^{-1} : \mathbf{R} - (-1, 1) \to \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$$

The graphs of $y = \text{cosec}\, x$ and $y = \text{cosec}^{-1} x$ are given in Fig 2.3 (i), (ii).



y = cosec *x*

Fig 2.3 (i)



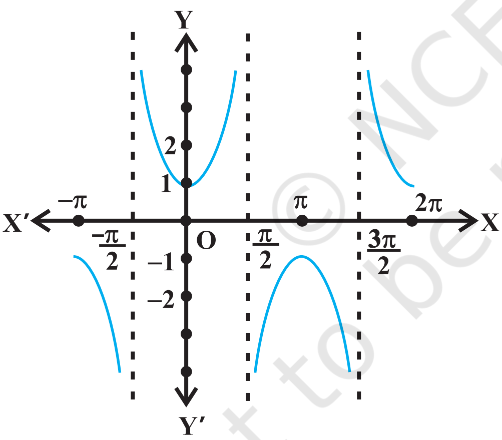
$y = \text{cosec}^{-1} x$

Fig 2.3 (ii)

Also, since $\sec x = \frac{1}{\cos x}$, the domain of $y = \sec x$ is the set $\mathbf{R} - \{x : x = (2n+1)\frac{\pi}{2}, n \in \mathbf{Z}\}$ and range is the set **R** – (–1, 1). It means that sec (secant function) assumes all real values except $-1 < y < 1$ and is not defined for odd multiples of $\frac{\pi}{2}$. If we restrict the domain of secant function to $[0, \pi] - \{\frac{\pi}{2}\}$, then it is one-one and onto with its range as the set $\mathbf{R} - (-1, 1)$. Actually, secant function restricted to any of the intervals $[-\pi, 0] - \{\frac{-\pi}{2}\}$, $[0, \pi] - \left\{\frac{\pi}{2}\right\}$, $[\pi, 2\pi] - \{\frac{3\pi}{2}\}$ etc., is bijective and its range is $\mathbf{R} - \{-1, 1\}$. Thus $\sec^{-1}$ can be defined as a function whose domain is $\mathbf{R} - (-1, 1)$ and range could be any of the intervals $[-\pi, 0] - \{\frac{-\pi}{2}\}$, $[0, \pi] - \{\frac{\pi}{2}\}$, $[\pi, 2\pi] - \{\frac{3\pi}{2}\}$ etc. Corresponding to each of these intervals, we get different branches of the function $\sec^{-1}$. The branch with range $[0, \pi] - \{\frac{\pi}{2}\}$ is called the *principal value branch* of the function $\sec^{-1}$. We thus have

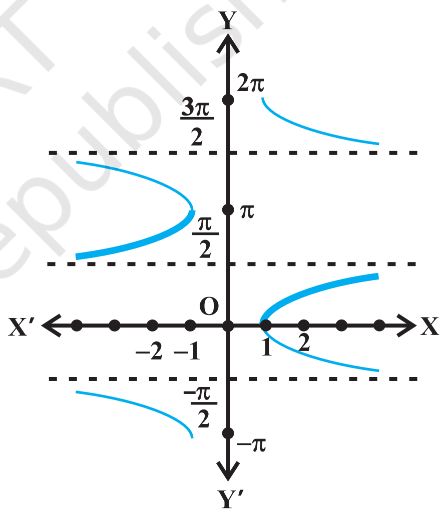
$$\sec^{-1} : \mathbf{R} - (-1,1) \rightarrow [0, \pi] - \{\frac{\pi}{2}\}$$

The graphs of the functions $y = \sec x$ and $y = \sec^{-1} x$ are given in Fig 2.4 (i), (ii).



$y = \sec x$

Fig 2.4 (i)



$y = \sec^{-1}x$

Fig 2.4 (ii)

Finally, we now discuss $\tan^{-1}$ and $\cot^{-1}$

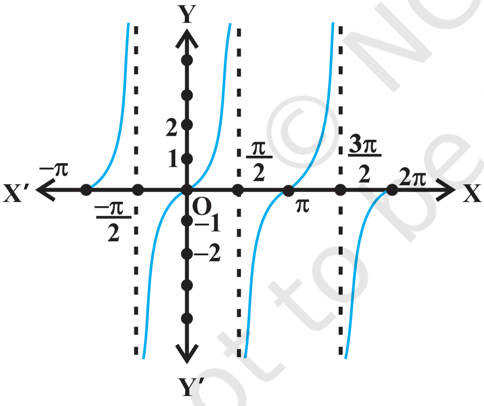
We know that the domain of the tan function (tangent function) is the set $\{x : x \in \mathbf{R} \text{ and } x \neq (2n+1)\frac{\pi}{2}, n \in \mathbf{Z}\}$ and the range is $\mathbf{R}$. It means that tan function is not defined for odd multiples of $\frac{\pi}{2}$. If we restrict the domain of tangent function to

$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, then it is one-one and onto with its range as **R**. Actually, tangent function restricted to any of the intervals $\left(\frac{-3\pi}{2}, \frac{-\pi}{2}\right)$, $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ etc., is bijective and its range is **R**. Thus $\tan^{-1}$ can be defined as a function whose domain is **R** and range could be any of the intervals $\left(\frac{-3\pi}{2}, \frac{-\pi}{2}\right)$, $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and so on. These intervals give different branches of the function $\tan^{-1}$. The branch with range $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ is called the *principal value branch* of the function $\tan^{-1}$.

We thus have

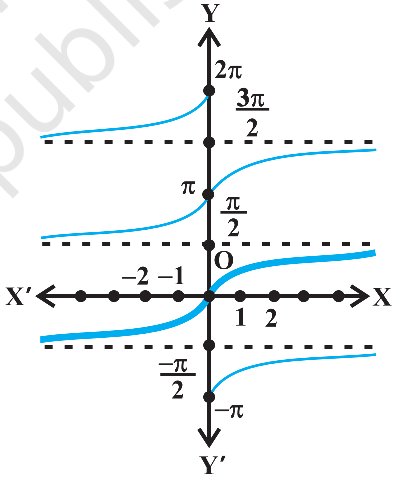
$$\tan^{-1} : \mathbf{R} \rightarrow \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

The graphs of the function $y = \tan x$ and $y = \tan^{-1}x$ are given in Fig 2.5 (i), (ii).



$y = \tan x$

Fig 2.5 (i)



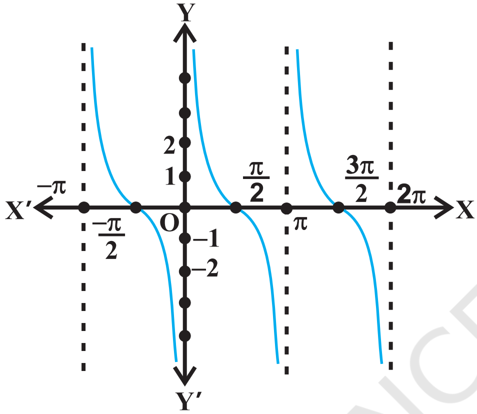
$y = \tan^{-1}x$

Fig 2.5 (ii)

We know that domain of the cot function (cotangent function) is the set $\{x : x \in \mathbf{R} \text{ and } x \neq n\pi, n \in \mathbf{Z}\}$ and range is **R**. It means that cotangent function is not defined for integral multiples of π. If we restrict the domain of cotangent function to (0, π), then it is bijective with and its range as **R**. In fact, cotangent function restricted to any of the intervals (–π, 0), (0, π), (π, 2π) etc., is bijective and its range is **R**. Thus $\cot^{-1}$ can be defined as a function whose domain is the **R** and range as any of the intervals $(-\pi, 0)$, $(0, \pi)$, $(\pi, 2\pi)$ etc. These intervals give different branches of the function $\cot^{-1}$. The function with range $(0, \pi)$ is called the *principal value branch* of the function $\cot^{-1}$. We thus have

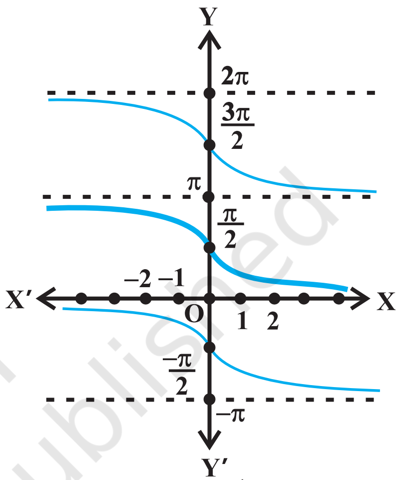
$$\cot^{-1} : \mathbf{R} \rightarrow (0, \pi)$$

The graphs of $y = \cot x$ and $y = \cot^{-1}x$ are given in Fig 2.6 (i), (ii).



$y = \cot x$

Fig 2.6 (i)



$y = \cot^{-1}x$

Fig 2.6 (ii)

The following table gives the inverse trigonometric function (principal value branches) along with their domains and ranges.

$\sin^{-1}$	:	$[-1, 1]$	$\rightarrow$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}$	:	$[-1, 1]$	$\rightarrow$	$[0, \pi]$
cosec^{-1}	:	$\mathbf{R} - (-1,1)$	$\rightarrow$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$\sec^{-1}$	:	$\mathbf{R} - (-1, 1)$	$\rightarrow$	$[0, \pi] - \{\frac{\pi}{2}\}$
$\tan^{-1}$	:	$\mathbf{R}$	$\rightarrow$	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1}$	:	$\mathbf{R}$	$\rightarrow$	$(0, \pi)$

Note

1. $\sin^{-1}x$ should not be confused with $(\sin x)^{-1}$. In fact $(\sin x)^{-1} = \frac{1}{\sin x}$ and similarly for other trigonometric functions.
2. Whenever no branch of an inverse trigonometric functions is mentioned, we mean the principal value branch of that function.
3. The value of an inverse trigonometric functions which lies in the range of principal branch is called the *principal value* of that inverse trigonometric functions.

We now consider some examples:

Example 1 Find the principal value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$.

Solution Let $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = y$. Then, $\sin y = \frac{1}{\sqrt{2}}$.

We know that the range of the principal value branch of $\sin^{-1}$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ and $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$. Therefore, principal value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is $\frac{\pi}{4}$

Example 2 Find the principal value of $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

Solution Let $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right) = y$. Then,

$$\cot y = \frac{-1}{\sqrt{3}} = -\cot\left(\frac{\pi}{3}\right) = \cot\left(\pi - \frac{\pi}{3}\right) = \cot\left(\frac{2\pi}{3}\right)$$

We know that the range of principal value branch of $\cot^{-1}$ is $(0, \pi)$ and $\cot\left(\frac{2\pi}{3}\right) = \frac{-1}{\sqrt{3}}$. Hence, principal value of $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$ is $\frac{2\pi}{3}$

EXERCISE 2.1

Find the principal values of the following:

1. $\sin^{-1}\left(-\frac{1}{2}\right)$ **2.** $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ **3.** $\text{cosec}^{-1}(2)$

4. $\tan^{-1}(-\sqrt{3})$ **5.** $\cos^{-1}\left(-\frac{1}{2}\right)$ **6.** $\tan^{-1}(-1)$

7. $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ **8.** $\cot^{-1}\ (\sqrt{3})$ **9.** $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

10. $\text{cosec}^{-1}\ (-\sqrt{2}\,)$

Find the values of the following:

11. $\tan^{-1}(1) + \cos^{-1}\ -\frac{1}{2}\ + \sin^{-1}\ -\frac{1}{2}$ **12.** $\cos^{-1}\ \frac{1}{2}\ + 2\sin^{-1}\ \frac{1}{2}$

13. If $\sin^{-1} x = y$, then

(A) $0 \le y \le \pi$ (B) $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(C) $0 < y < \pi$ (D) $-\frac{\pi}{2} < y < \frac{\pi}{2}$

14. $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$ is equal to

(A) π (B) $-\frac{\pi}{3}$ (C) $\frac{\pi}{3}$ (D) $\frac{2\pi}{3}$

2.3 Properties of Inverse Trigonometric Functions

In this section, we shall prove some important properties of inverse trigonometric functions. It may be mentioned here that these results are valid within the principal value branches of the corresponding inverse trigonometric functions and wherever they are defined. Some results may not be valid for all values of the domains of inverse trigonometric functions. In fact, they will be valid only for some values of x for which inverse trigonometric functions are defined. We will not go into the details of these values of x in the domain as this discussion goes beyond the scope of this textbook.

Let us recall that if $y = \sin^{-1}x$, then $x = \sin y$ and if $x = \sin y$, then $y = \sin^{-1}x$. This is equivalent to

$$\sin(\sin^{-1} x) = x,\ x \in [-1, 1] \text{ and } \sin^{-1}(\sin x) = x,\ x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

For suitable values of domain similar results follow for remaining trigonometric functions.

We now consider some examples.

Example 3 Show that

(i) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\sin^{-1}x,\ -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

(ii) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\cos^{-1}x,\ \frac{1}{\sqrt{2}} \le x \le 1$

Solution

(i) Let $x = \sin\theta$. Then $\sin^{-1}x = \theta$. We have

$$\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$$

$$= \sin^{-1}(2\sin\theta\cos\theta) = \sin^{-1}(\sin 2\theta) = 2\theta$$

$$= 2\sin^{-1}x$$

(ii) Take $x = \cos\theta$, then proceeding as above, we get, $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\cos^{-1}x$

Example 4 Express $\tan^{-1}\frac{\cos x}{1-\sin x},\ \frac{-3\pi}{2} < x < \frac{\pi}{2}$ in the simplest form.

Solution We write

$$\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right) = \tan^{-1}\left[\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}\right]$$

$$= \tan^{-1}\left[\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}\right]$$

$$= \tan^{-1}\left[\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right] = \tan^{-1}\left[\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right] = \frac{\pi}{4}+\frac{x}{2}$$

Example 5 Write $\cot^{-1}\left(\dfrac{1}{\sqrt{x^2-1}}\right)$, $x > 1$ in the simplest form.

Solution Let $x = \sec\theta$, then $\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta$

Therefore, $\cot^{-1}\dfrac{1}{\sqrt{x^2-1}} = \cot^{-1}(\cot\theta) = \theta = \sec^{-1}x$, which is the simplest form.

EXERCISE 2.2

Prove the following:

1. $3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$, $x \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$

2. $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$, $x \in \left[\dfrac{1}{2}, 1\right]$

Write the following functions in the simplest form:

3. $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}$, $x \neq 0$

4. $\tan^{-1}\left(\sqrt{\dfrac{1-\cos x}{1+\cos x}}\right)$, $0 < x < \pi$

5. $\tan^{-1}\left(\dfrac{\cos x - \sin x}{\cos x + \sin x}\right)$, $\dfrac{-\pi}{4} < x < \dfrac{3\pi}{4}$

6. $\tan^{-1}\dfrac{x}{\sqrt{a^2-x^2}}$, $|x| < a$

7. $\tan^{-1}\left(\dfrac{3a^2x - x^3}{a^3 - 3ax^2}\right)$, $a > 0$; $\dfrac{-a}{\sqrt{3}} < x < \dfrac{a}{\sqrt{3}}$

Find the values of each of the following:

8. $\tan^{-1}\left[2\cos\left(2\sin^{-1}\dfrac{1}{2}\right)\right]$

9. $\tan\dfrac{1}{2}\left[\sin^{-1}\dfrac{2x}{1+x^2} + \cos^{-1}\dfrac{1-y^2}{1+y^2}\right]$, $|x| < 1$, $y > 0$ and $xy < 1$

Find the values of each of the expressions in Exercises 16 to 18.

10. $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

11. $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

12. $\tan\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)$

13. $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ is equal to

(A) $\frac{7\pi}{6}$ (B) $\frac{5\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{6}$

14. $\sin\left(\frac{\pi}{3}-\sin^{-1}(-\frac{1}{2})\right)$ is equal to

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) 1

15. $\tan^{-1}\sqrt{3}-\cot^{-1}(-\sqrt{3})$ is equal to

(A) π (B) $-\frac{\pi}{2}$ (C) 0 (D) $2\sqrt{3}$

Miscellaneous Examples

Example 6 Find the value of $\sin^{-1}(\sin\frac{3\pi}{5})$

Solution We know that $\sin^{-1}(\sin x) = x$. Therefore, $\sin^{-1}(\sin\frac{3\pi}{5}) = \frac{3\pi}{5}$

But $\frac{3\pi}{5} \notin \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, which is the principal branch of $\sin^{-1} x$

However $\sin(\frac{3\pi}{5}) = \sin(\pi-\frac{3\pi}{5}) = \sin\frac{2\pi}{5}$ and $\frac{2\pi}{5} \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Therefore $\sin^{-1}(\sin\frac{3\pi}{5}) = \sin^{-1}(\sin\frac{2\pi}{5}) = \frac{2\pi}{5}$

Miscellaneous Exercise on Chapter 2

Find the value of the following:

1. $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$

2. $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$

Prove that

3. $2\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{24}{7}$

4. $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{77}{36}$

5. $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$

6. $\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{56}{65}$

7. $\tan^{-1}\frac{63}{16} = \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5}$

Prove that

8. $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\frac{1-x}{1+x}$, $x \in [0, 1]$

9. $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}$, $x \in \left(0, \frac{\pi}{4}\right)$

10. $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x$, $-\frac{1}{\sqrt{2}} \le x \le 1$ [Hint: Put $x = \cos 2\theta$]

Solve the following equations:

11. $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$

12. $\tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}\tan^{-1}x, (x > 0)$

13. $\sin(\tan^{-1}x)$, $|x| < 1$ is equal to

(A) $\frac{x}{\sqrt{1-x^2}}$ (B) $\frac{1}{\sqrt{1-x^2}}$ (C) $\frac{1}{\sqrt{1+x^2}}$ (D) $\frac{x}{\sqrt{1+x^2}}$

14. $\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$, then x is equal to

(A) $0, \frac{1}{2}$ (B) $1, \frac{1}{2}$ (C) 0 (D) $\frac{1}{2}$

Summary

- The domains and ranges (principal value branches) of inverse trigonometric functions are given in the following table:

Functions	Domain	Range (Principal Value Branches)
$y = \sin^{-1} x$	[–1, 1]	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1} x$	[–1, 1]	$[0, \pi]$
$y = \text{cosec}^{-1} x$	**R** – (–1,1)	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$y = \sec^{-1} x$	**R** – (–1, 1)	$[0, \pi] - \{\frac{\pi}{2}\}$
$y = \tan^{-1} x$	**R**	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \cot^{-1} x$	**R**	$(0, \pi)$

- $\sin^{-1}x$ should not be confused with $(\sin x)^{-1}$. In fact $(\sin x)^{-1} = \frac{1}{\sin x}$ and similarly for other trigonometric functions.
- The value of an inverse trigonometric functions which lies in its principal value branch is called the *principal value* of that inverse trigonometric functions.

For suitable values of domain, we have

- $y = \sin^{-1} x \Rightarrow x = \sin y$
- $x = \sin y \Rightarrow y = \sin^{-1} x$
- $\sin(\sin^{-1} x) = x$
- $\sin^{-1}(\sin x) = x$

Historical Note

The study of trigonometry was first started in India. The ancient Indian Mathematicians, Aryabhata (476A.D.), Brahmagupta (598 A.D.), Bhaskara I (600 A.D.) and Bhaskara II (1114 A.D.) got important results of trigonometry. All this knowledge went from India to Arabia and then from there to Europe. The Greeks had also started the study of trigonometry but their approach was so clumsy that when the Indian approach became known, it was immediately adopted throughout the world.

In India, the predecessor of the modern trigonometric functions, known as the sine of an angle, and the introduction of the sine function represents one of the main contribution of the *siddhantas* (Sanskrit astronomical works) to mathematics.

Bhaskara I (about 600 A.D.) gave formulae to find the values of sine functions for angles more than 90°. A sixteenth century Malayalam work *Yuktibhasa* contains a proof for the expansion of sin (A + B). Exact expression for sines or cosines of 18°, 36°, 54°, 72°, etc., were given by Bhaskara II.

The symbols $\sin^{-1} x$, $\cos^{-1} x$, etc., for arc sin x, arc cos x, etc., were suggested by the astronomer Sir John F.W. Hersehel (1813) The name of Thales (about 600 B.C.) is invariably associated with height and distance problems. He is credited with the determination of the height of a great pyramid in Egypt by measuring shadows of the pyramid and an auxiliary staff (or gnomon) of known height, and comparing the ratios:

$$\frac{\mathrm{H}}{\mathrm{S}} = \frac{h}{s} = \tan \text{(sun's altitude)}$$

Thales is also said to have calculated the distance of a ship at sea through the proportionality of sides of similar triangles. Problems on height and distance using the similarity property are also found in ancient Indian works.

Chapter 3

MATRICES

❖ *The essence of Mathematics lies in its freedom. — CANTOR* ❖

3.1 Introduction

The knowledge of matrices is necessary in various branches of mathematics. Matrices are one of the most powerful tools in mathematics. This mathematical tool simplifies our work to a great extent when compared with other straight forward methods. The evolution of concept of matrices is the result of an attempt to obtain compact and simple methods of solving system of linear equations. Matrices are not only used as a representation of the coefficients in system of linear equations, but utility of matrices far exceeds that use. Matrix notation and operations are used in electronic spreadsheet programs for personal computer, which in turn is used in different areas of business and science like budgeting, sales projection, cost estimation, analysing the results of an experiment etc. Also, many physical operations such as magnification, rotation and reflection through a plane can be represented mathematically by matrices. Matrices are also used in cryptography. This mathematical tool is not only used in certain branches of sciences, but also in genetics, economics, sociology, modern psychology and industrial management.

In this chapter, we shall find it interesting to become acquainted with the fundamentals of matrix and matrix algebra.

3.2 Matrix

Suppose we wish to express the information that Radha has 15 notebooks. We may express it as [15] with the understanding that the number inside [ ] is the number of notebooks that Radha has. Now, if we have to express that Radha has 15 notebooks and 6 pens. We may express it as [15 6] with the understanding that first number inside [ ] is the number of notebooks while the other one is the number of pens possessed by Radha. Let us now suppose that we wish to express the information of possession of notebooks and pens by Radha and her two friends Fauzia and Simran which is as follows:

Radha	has	15	notebooks	and	6 pens,
Fauzia	has	10	notebooks	and	2 pens,
Simran	has	13	notebooks	and	5 pens.

Now this could be arranged in the tabular form as follows:

	Notebooks	**Pens**
Radha	15	6
Fauzia	10	2
Simran	13	5

and this can be expressed as

$$\begin{bmatrix} 15 & 6 \\ 10 & 2 \\ 13 & 5 \end{bmatrix} \begin{matrix} \leftarrow \text{First row} \\ \leftarrow \text{Second row} \\ \leftarrow \text{Third row} \end{matrix}$$

$$\begin{matrix} \uparrow & \uparrow \\ \text{First Column} & \text{Second Column} \end{matrix}$$

or

	Radha	**Fauzia**	**Simran**
Notebooks	15	10	13
Pens	6	2	5

which can be expressed as:

$$\begin{bmatrix} 15 & 10 & 13 \\ 6 & 2 & 5 \end{bmatrix} \begin{matrix} \leftarrow \text{First row} \\ \leftarrow \text{Second row} \end{matrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \text{First Column} & \text{Second Column} & \text{Third Column} \end{matrix}$$

In the first arrangement the entries in the first column represent the number of note books possessed by Radha, Fauzia and Simran, respectively and the entries in the second column represent the number of pens possessed by Radha, Fauzia and Simran, respectively. Similarly, in the second arrangement, the entries in the first row represent the number of notebooks possessed by Radha, Fauzia and Simran, respectively. The entries in the second row represent the number of pens possessed by Radha, Fauzia and Simran, respectively. An arrangement or display of the above kind is called a *matrix*. Formally, we define matrix as:

Definition 1 A *matrix* is an ordered rectangular array of numbers or functions. The numbers or functions are called the elements or the entries of the matrix.

We denote matrices by capital letters. The following are some examples of matrices:

$$A = \begin{bmatrix} -2 & 5 \\ 0 & \sqrt{5} \\ 3 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 2+i & 3 & -\frac{1}{2} \\ 3.5 & -1 & 2 \\ \sqrt{3} & 5 & \frac{5}{7} \end{bmatrix}, \quad C = \begin{bmatrix} 1+x & x^3 & 3 \\ \cos x & \sin x + 2 & \tan x \end{bmatrix}$$

In the above examples, the horizontal lines of elements are said to constitute, ***rows*** of the matrix and the vertical lines of elements are said to constitute, ***columns*** of the matrix. Thus A has 3 rows and 2 columns, B has 3 rows and 3 columns while C has 2 rows and 3 columns.

3.2.1 *Order of a matrix*

A matrix having m rows and n columns is called a matrix of *order* $m \times n$ or simply $m \times n$ matrix (read as an m by n matrix). So referring to the above examples of matrices, we have A as 3×2 matrix, B as 3×3 matrix and C as 2×3 matrix. We observe that A has $3 \times 2 = 6$ elements, B and C have 9 and 6 elements, respectively.

In general, an $m \times n$ matrix has the following rectangular array:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1j} \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots & a_{2j} \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & a_{i3} \cdots & a_{ij} \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} \cdots & a_{mj} \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

or $A = [a_{ij}]_{m \times n}$, $1 \le i \le m$, $1 \le j \le n$ $i, j \in N$

Thus the i^{th} row consists of the elements $a_{i1}, a_{i2}, a_{i3}, \ldots, a_{in}$, while the j^{th} column consists of the elements $a_{1j}, a_{2j}, a_{3j}, \ldots, a_{mj}$,

In general a_{ij}, is an element lying in the i^{th} row and j^{th} column. We can also call it as the $(i, j)^{th}$ element of A. The number of elements in an $m \times n$ matrix will be equal to mn.

Note In this chapter

1. We shall follow the notation, namely $A = [a_{ij}]_{m \times n}$ to indicate that A is a matrix of order $m \times n$.
2. We shall consider only those matrices whose elements are real numbers or functions taking real values.

We can also represent any point (x, y) in a plane by a matrix (column or row) as $\begin{bmatrix} x \\ y \end{bmatrix}$ (or $[x, y]$). For example point P(0, 1) as a matrix representation may be given as

$$P = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ or } [0\ 1].$$

Observe that in this way we can also express the vertices of a closed rectilinear figure in the form of a matrix. For example, consider a quadrilateral ABCD with vertices A (1, 0), B (3, 2), C (1, 3), D (–1, 2).

Now, quadrilateral ABCD in the matrix form, can be represented as

$$X = \begin{matrix} \begin{matrix} A & B & C & D \end{matrix} \\ \begin{bmatrix} 1 & 3 & 1 & -1 \\ 0 & 2 & 3 & 2 \end{bmatrix}_{2 \times 4} \end{matrix} \quad \text{or} \quad Y = \begin{matrix} A \\ B \\ C \\ D \end{matrix} \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 1 & 3 \\ -1 & 2 \end{bmatrix}_{4 \times 2}$$

Thus, matrices can be used as representation of vertices of geometrical figures in a plane.

Now, let us consider some examples.

Example 1 Consider the following information regarding the number of men and women workers in three factories I, II and III

	Men workers	Women workers
I	30	25
II	25	31
III	27	26

Represent the above information in the form of a 3×2 matrix. What does the entry in the third row and second column represent?

Solution The information is represented in the form of a 3 × 2 matrix as follows:

$$A = \begin{bmatrix} 30 & 25 \\ 25 & 31 \\ 27 & 26 \end{bmatrix}$$

The entry in the third row and second column represents the number of women workers in factory III.

Example 2 If a matrix has 8 elements, what are the possible orders it can have?

Solution We know that if a matrix is of order $m \times n$, it has mn elements. Thus, to find all possible orders of a matrix with 8 elements, we will find all ordered pairs of natural numbers, whose product is 8.

Thus, all possible ordered pairs are (1, 8), (8, 1), (4, 2), (2, 4)

Hence, possible orders are 1 × 8, 8 ×1, 4 × 2, 2 × 4

Example 3 Construct a 3 × 2 matrix whose elements are given by $a_{ij} = \frac{1}{2}|i - 3j|$.

Solution In general a 3 × 2 matrix is given by $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$.

Now $\quad a_{ij} = \frac{1}{2}|i - 3j|$, $i = 1, 2, 3$ and $j = 1, 2$.

Therefore

$$a_{11} = \frac{1}{2}|1 - 3 \times 1| = 1 \qquad a_{12} = \frac{1}{2}|1 - 3 \times 2| = \frac{5}{2}$$

$$a_{21} = \frac{1}{2}|2 - 3 \times 1| = \frac{1}{2} \qquad a_{22} = \frac{1}{2}|2 - 3 \times 2| = 2$$

$$a_{31} = \frac{1}{2}|3 - 3 \times 1| = 0 \qquad a_{32} = \frac{1}{2}|3 - 3 \times 2| = \frac{3}{2}$$

Hence the required matrix is given by $A = \begin{bmatrix} 1 & \frac{5}{2} \\ \frac{1}{2} & 2 \\ 0 & \frac{3}{2} \end{bmatrix}$.

3.3 Types of Matrices

In this section, we shall discuss different types of matrices.

(i) **Column matrix**

A matrix is said to be a *column matrix* if it has only one column.

For example, $A = \begin{bmatrix} 0 \\ \sqrt{3} \\ -1 \\ 1/2 \end{bmatrix}$ is a column matrix of order 4 × 1.

In general, $A = [a_{ij}]_{m \times 1}$ is a column matrix of order $m \times 1$.

(ii) **Row matrix**

A matrix is said to be a *row matrix* if it has only one row.

For example, $B = \begin{bmatrix} -\frac{1}{2} & \sqrt{5} & 2 & 3 \end{bmatrix}_{1\times 4}$ is a row matrix.

In general, $B = [b_{ij}]_{1 \times n}$ is a row matrix of order $1 \times n$.

(iii) **Square matrix**

A matrix in which the number of rows are equal to the number of columns, is said to be a *square matrix*. Thus an $m \times n$ matrix is said to be a square matrix if $m = n$ and is known as a square matrix of order 'n'.

For example $A = \begin{bmatrix} 3 & -1 & 0 \\ \frac{3}{2} & 3\sqrt{2} & 1 \\ 4 & 3 & -1 \end{bmatrix}$ is a square matrix of order 3.

In general, $A = [a_{ij}]_{m \times m}$ is a square matrix of order m.

Note If $A = [a_{ij}]$ is a square matrix of order n, then elements (entries) $a_{11}, a_{22}, ..., a_{nn}$ are said to constitute the *diagonal*, of the matrix A. Thus, if $A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & 4 & -1 \\ 3 & 5 & 6 \end{bmatrix}$.

Then the elements of the diagonal of A are 1, 4, 6.

(iv) **Diagonal matrix**

A square matrix $B = [b_{ij}]_{m \times m}$ is said to be a *diagonal matrix* if all its non diagonal elements are zero, that is a matrix $B = [b_{ij}]_{m \times m}$ is said to be a diagonal matrix if $b_{ij} = 0$, when $i \neq j$.

For example, $A = [4]$, $B = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$, $C = \begin{bmatrix} -1.1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, are diagonal matrices of order 1, 2, 3, respectively.

(v) **Scalar matrix**

A diagonal matrix is said to be a *scalar matrix* if its diagonal elements are equal, that is, a square matrix $B = [b_{ij}]_{n \times n}$ is said to be a scalar matrix if

$$b_{ij} = 0, \quad \text{when } i \neq j$$
$$b_{ij} = k, \quad \text{when } i = j, \text{ for some constant } k.$$

For example

$$A = [3], \quad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$$

are scalar matrices of order 1, 2 and 3, respectively.

(vi) **Identity matrix**

A square matrix in which elements in the diagonal are all 1 and rest are all zero is called an *identity matrix*. In other words, the square matrix $A = [a_{ij}]_{n \times n}$ is an identity matrix, if $a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

We denote the identity matrix of order n by I_n. When order is clear from the context, we simply write it as I.

For example $[1]$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are identity matrices of order 1, 2 and 3, respectively.

Observe that a scalar matrix is an identity matrix when $k = 1$. But every identity matrix is clearly a scalar matrix.

(vii) **Zero matrix**

A matrix is said to be *zero matrix* or *null matrix* if all its elements are zero.

For example, [0], $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, [0, 0] are all zero matrices. We denote zero matrix by O. Its order will be clear from the context.

3.3.1 *Equality of matrices*

Definition 2 Two matrices A = $[a_{ij}]$ and B = $[b_{ij}]$ are said to be equal if

(i) they are of the same order

(ii) each element of A is equal to the corresponding element of B, that is $a_{ij} = b_{ij}$ for all i and j.

For example, $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ are equal matrices but $\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ are not equal matrices. Symbolically, if two matrices A and B are equal, we write A = B.

If $\begin{bmatrix} x & y \\ z & a \\ b & c \end{bmatrix} = \begin{bmatrix} -1.5 & 0 \\ 2 & \sqrt{6} \\ 3 & 2 \end{bmatrix}$, then $x = -1.5, y = 0, z = 2, a = \sqrt{6}, b = 3, c = 2$

Example 4 If $\begin{bmatrix} x+3 & z+4 & 2y-7 \\ -6 & a-1 & 0 \\ b-3 & -21 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 3y-2 \\ -6 & -3 & 2c+2 \\ 2b+4 & -21 & 0 \end{bmatrix}$

Find the values of a, b, c, x, y and z.

Solution As the given matrices are equal, therefore, their corresponding elements must be equal. Comparing the corresponding elements, we get

$$x + 3 = 0, \qquad z + 4 = 6, \qquad 2y - 7 = 3y - 2$$

$$a - 1 = -3, \qquad 0 = 2c + 2 \qquad b - 3 = 2b + 4,$$

Simplifying, we get

$$a = -2, b = -7, c = -1, x = -3, y = -5, z = 2$$

Example 5 Find the values of a, b, c, and d from the following equation:

$$\begin{bmatrix} 2a+b & a-2b \\ 5c-d & 4c+3d \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 11 & 24 \end{bmatrix}$$

Solution By equality of two matrices, equating the corresponding elements, we get

$$2a + b = 4 \qquad 5c - d = 11$$
$$a - 2b = -3 \qquad 4c + 3d = 24$$

Solving these equations, we get

$$a = 1,\ b = 2,\ c = 3 \text{ and } d = 4$$

EXERCISE 3.1

1. In the matrix $A = \begin{bmatrix} 2 & 5 & 19 & -7 \\ 35 & -2 & \frac{5}{2} & 12 \\ \sqrt{3} & 1 & -5 & 17 \end{bmatrix}$, write:

(i) The order of the matrix, (ii) The number of elements,

(iii) Write the elements $a_{13}, a_{21}, a_{33}, a_{24}, a_{23}$.

2. If a matrix has 24 elements, what are the possible orders it can have? What, if it has 13 elements?

3. If a matrix has 18 elements, what are the possible orders it can have? What, if it has 5 elements?

4. Construct a 2 × 2 matrix, $A = [a_{ij}]$, whose elements are given by:

(i) $a_{ij} = \frac{(i+j)^2}{2}$ (ii) $a_{ij} = \frac{i}{j}$ (iii) $a_{ij} = \frac{(i+2j)^2}{2}$

5. Construct a 3 × 4 matrix, whose elements are given by:

(i) $a_{ij} = \frac{1}{2}|-3i + j|$ (ii) $a_{ij} = 2i - j$

6. Find the values of x, y and z from the following equations:

(i) $\begin{bmatrix} 4 & 3 \\ x & 5 \end{bmatrix} = \begin{bmatrix} y & z \\ 1 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} x+y & 2 \\ 5+z & xy \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$ (iii) $\begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$

7. Find the value of a, b, c and d from the equation:

$$\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$$

8. $A = [a_{ij}]_{m \times n}$ is a square matrix, if

(A) $m < n$ (B) $m > n$ (C) $m = n$ (D) None of these

9. Which of the given values of x and y make the following pair of matrices equal

$$\begin{bmatrix} 3x+7 & 5 \\ y+1 & 2-3x \end{bmatrix}, \begin{bmatrix} 0 & y-2 \\ 8 & 4 \end{bmatrix}$$

(A) $x = \frac{-1}{3}, \quad y = 7$ (B) Not possible to find

(C) $y = 7, \quad x = \frac{-2}{3}$ (D) $x = \frac{-1}{3}, \quad y = \frac{-2}{3}$

10. The number of all possible matrices of order 3×3 with each entry 0 or 1 is:

(A) 27 (B) 18 (C) 81 (D) 512

3.4 Operations on Matrices

In this section, we shall introduce certain operations on matrices, namely, addition of matrices, multiplication of a matrix by a scalar, difference and multiplication of matrices.

3.4.1 *Addition of matrices*

Suppose Fatima has two factories at places A and B. Each factory produces sport shoes for boys and girls in three different price categories labelled 1, 2 and 3. The quantities produced by each factory are represented as matrices given below:

	Factory at A			Factory at B	
	Boys	Girls		Boys	Girls
1	80	60	1	90	50
2	75	65	2	70	55
3	90	85	3	75	75

Suppose Fatima wants to know the total production of sport shoes in each price category. Then the total production

In category 1 : for boys (80 + 90), for girls (60 + 50)

In category 2 : for boys (75 + 70), for girls (65 + 55)

In category 3 : for boys (90 + 75), for girls (85 + 75)

This can be represented in the matrix form as $\begin{bmatrix} 80+90 & 60+50 \\ 75+70 & 65+55 \\ 90+75 & 85+75 \end{bmatrix}$.

This new matrix is the **sum** of the above two matrices. We observe that the sum of two matrices is a matrix obtained by adding the corresponding elements of the given matrices. Furthermore, the two matrices have to be of the same order.

Thus, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ is a 2×3 matrix and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$ is another 2×3 matrix. Then, we define $A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix}$.

In general, if $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices of the same order, say $m \times n$. Then, the sum of the two matrices A and B is *defined* as a matrix $C = [c_{ij}]_{m \times n}$, where $c_{ij} = a_{ij} + b_{ij}$, for all possible values of i and j.

Example 6 Given $A = \begin{bmatrix} \sqrt{3} & 1 & -1 \\ 2 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & \sqrt{5} & 1 \\ -2 & 3 & \frac{1}{2} \end{bmatrix}$, find A + B

Since A, B are of the same order 2×3. Therefore, addition of A and B is defined and is given by

$$A + B = \begin{bmatrix} 2+\sqrt{3} & 1+\sqrt{5} & 1-1 \\ 2-2 & 3+3 & 0+\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2+\sqrt{3} & 1+\sqrt{5} & 0 \\ 0 & 6 & \frac{1}{2} \end{bmatrix}$$

Note

1. We emphasise that if A and B are not of the same order, then A + B is not defined. For example if $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix}$, then A + B is not defined.
2. We may observe that addition of matrices is an example of binary operation on the set of matrices of the same order.

3.4.2 *Multiplication of a matrix by a scalar*

Now suppose that Fatima has doubled the production at a factory A in all categories (refer to 3.4.1).

Previously quantities (in standard units) produced by factory A were

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \text{Boys} \quad \text{Girls} \\ \begin{bmatrix} 80 & 60 \\ 75 & 65 \\ 90 & 85 \end{bmatrix} \end{array}$$

Revised quantities produced by factory A are as given below:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \text{Boys} \quad \text{Girls} \\ \begin{bmatrix} 2\times 80 & 2\times 60 \\ 2\times 75 & 2\times 65 \\ 2\times 90 & 2\times 85 \end{bmatrix} \end{array}$$

This can be represented in the matrix form as $\begin{bmatrix} 160 & 120 \\ 150 & 130 \\ 180 & 170 \end{bmatrix}$. We observe that the new matrix is obtained by multiplying each element of the previous matrix by 2.

In general, we may define *multiplication of a matrix* by a scalar as follows: if $A = [a_{ij}]_{m \times n}$ is a matrix and k is a scalar, then kA is another matrix which is obtained by multiplying each element of A by the scalar k.

In other words, $kA = k[a_{ij}]_{m \times n} = [k(a_{ij})]_{m \times n}$, that is, $(i, j)^{th}$ element of kA is ka_{ij} for all possible values of i and j.

For example, if $A = \begin{bmatrix} 3 & 1 & 1.5 \\ \sqrt{5} & 7 & -3 \\ 2 & 0 & 5 \end{bmatrix}$, then

$$3A = 3\begin{bmatrix} 3 & 1 & 1.5 \\ \sqrt{5} & 7 & -3 \\ 2 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 3 & 4.5 \\ 3\sqrt{5} & 21 & -9 \\ 6 & 0 & 15 \end{bmatrix}$$

Negative of a matrix The negative of a matrix is denoted by – A. We define – A = (– 1) A.

For example, let $A = \begin{bmatrix} 3 & 1 \\ -5 & x \end{bmatrix}$, then – A is given by

$$-A = (-1)A = (-1)\begin{bmatrix} 3 & 1 \\ -5 & x \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 5 & -x \end{bmatrix}$$

Difference of matrices If A = $[a_{ij}]$, B = $[b_{ij}]$ are two matrices of the same order, say $m \times n$, then difference A – B is defined as a matrix D = $[d_{ij}]$, where $d_{ij} = a_{ij} - b_{ij}$, for all value of i and j. In other words, D = A – B = A + (–1) B, that is sum of the matrix A and the matrix – B.

Example 7 If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$, then find 2A – B.

Solution We have

$$2A - B = 2\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & 6 \\ 4 & 6 & 2 \end{bmatrix} + \begin{bmatrix} -3 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2-3 & 4+1 & 6-3 \\ 4+1 & 6+0 & 2-2 \end{bmatrix} = \begin{bmatrix} -1 & 5 & 3 \\ 5 & 6 & 0 \end{bmatrix}$$

3.4.3 *Properties of matrix addition*

The addition of matrices satisfy the following properties:

(i) **Commutative Law** If A = $[a_{ij}]$, B = $[b_{ij}]$ are matrices of the same order, say $m \times n$, then A + B = B + A.

Now A + B = $[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$
$= [b_{ij} + a_{ij}]$ (addition of numbers is commutative)
$= ([b_{ij}] + [a_{ij}])$ = B + A

(ii) **Associative Law** For any three matrices A = $[a_{ij}]$, B = $[b_{ij}]$, C = $[c_{ij}]$ of the same order, say $m \times n$, (A + B) + C = A + (B + C).

Now (A + B) + C = $([a_{ij}] + [b_{ij}]) + [c_{ij}]$
$= [a_{ij} + b_{ij}] + [c_{ij}] = [(a_{ij} + b_{ij}) + c_{ij}]$
$= [a_{ij} + (b_{ij} + c_{ij})]$ (Why?)
$= [a_{ij}] + [(b_{ij} + c_{ij})] = [a_{ij}] + ([b_{ij}] + [c_{ij}])$ = A + (B + C)

(iii) **Existence of additive identity** Let A = $[a_{ij}]$ be an $m \times n$ matrix and O be an $m \times n$ zero matrix, then A + O = O + A = A. In other words, O is the additive identity for matrix addition.

(iv) **The existence of additive inverse** Let A = $[a_{ij}]_{m \times n}$ be any matrix, then we have another matrix as – A = $[- a_{ij}]_{m \times n}$ such that A + (– A) = (– A) + A= O. So – A is the additive inverse of A or negative of A.

3.4.4 *Properties of scalar multiplication of a matrix*

If A = $[a_{ij}]$ and B = $[b_{ij}]$ be two matrices of the same order, say $m \times n$, and k and l are scalars, then

(i) $k(A + B) = k\,A + kB$, (ii) $(k + l)A = k\,A + l\,A$

(ii) $k\,(A + B) = k\,([a_{ij}] + [b_{ij}])$

$$= k\,[a_{ij} + b_{ij}] = [k\,(a_{ij} + b_{ij})] = [(k\,a_{ij}) + (k\,b_{ij})]$$

$$= [k\,a_{ij}] + [k\,b_{ij}] = k\,[a_{ij}] + k\,[b_{ij}] = kA + kB$$

(iii) $(k + l)\,A = (k + l)\,[a_{ij}]$

$$= [(k + l)\,a_{ij}] + [k\,a_{ij}] + [l\,a_{ij}] = k\,[a_{ij}] + l\,[a_{ij}] = k\,A + l\,A$$

Example 8 If $A = \begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix}$, then find the matrix X, such that 2A + 3X = 5B.

Solution We have 2A + 3X = 5B

or $2A + 3X - 2A = 5B - 2A$

or $2A - 2A + 3X = 5B - 2A$ (Matrix addition is commutative)

or $O + 3X = 5B - 2A$ (– 2A is the additive inverse of 2A)

or $3X = 5B - 2A$ (O is the additive identity)

or $X = \frac{1}{3}(5B - 2A)$

or $$X = \frac{1}{3}\left(5\begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix} - 2\begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}\right) = \frac{1}{3}\left(\begin{bmatrix} 10 & -10 \\ 20 & 10 \\ -25 & 5 \end{bmatrix} + \begin{bmatrix} -16 & 0 \\ -8 & 4 \\ -6 & -12 \end{bmatrix}\right)$$

$$= \frac{1}{3}\begin{bmatrix} 10-16 & -10+0 \\ 20-8 & 10+4 \\ -25-6 & 5-12 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -6 & -10 \\ 12 & 14 \\ -31 & -7 \end{bmatrix} = \begin{bmatrix} -2 & \frac{-10}{3} \\ 4 & \frac{14}{3} \\ \frac{-31}{3} & \frac{-7}{3} \end{bmatrix}$$

Example 9 Find X and Y, if $X + Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$.

Solution We have $(X+Y)+(X-Y) = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$.

or $$(X + X) + (Y - Y) = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} \Rightarrow 2X = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix}$$

or $$X = \frac{1}{2}\begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$$

Also $$(X + Y) - (X - Y) = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$$

or $$(X - X) + (Y + Y) = \begin{bmatrix} 5-3 & 2-6 \\ 0 & 9+1 \end{bmatrix} \Rightarrow 2Y = \begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix}$$

or $$Y = \frac{1}{2}\begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$$

Example 10 Find the values of x and y from the following equation:

$$2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

Solution We have

$$2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x & 10 \\ 14 & 2y-6 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

or $\begin{bmatrix} 2x+3 & 10-4 \\ 14+1 & 2y-6+2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x+3 & 6 \\ 15 & 2y-4 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$

or $2x + 3 = 7$ and $2y - 4 = 14$ (Why?)

or $2x = 7 - 3$ and $2y = 18$

or $x = \frac{4}{2}$ and $y = \frac{18}{2}$

i.e. $x = 2$ and $y = 9$.

Example 11 Two farmers Ramkishan and Gurcharan Singh cultivates only three varieties of rice namely Basmati, Permal and Naura. The sale (in Rupees) of these varieties of rice by both the farmers in the month of September and October are given by the following matrices A and B.

September Sales (in Rupees)

	Basmati	Permal	Naura	
A =	10,000	20,000	30,000	Ramkishan
	50,000	30,000	10,000	Gurcharan Singh

October Sales (in Rupees)

	Basmati	Permal	Naura	
B =	5000	10,000	6000	Ramkishan
	20,000	10,000	10,000	Gurcharan Singh

(i) Find the combined sales in September and October for each farmer in each variety.

(ii) Find the decrease in sales from September to October.

(iii) If both farmers receive 2% profit on gross sales, compute the profit for each farmer and for each variety sold in October.

Solution

(i) Combined sales in September and October for each farmer in each variety is given by

	Basmati	Permal	Naura	
A + B =	15,000	30,000	36,000	Ramkishan
	70,000	40,000	20,000	Gurcharan Singh

(ii) Change in sales from September to October is given by

$$A - B = \begin{array}{c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \left[\begin{array}{ccc} 5000 & 10,000 & 24,000 \\ 30,000 & 20,000 & 0 \end{array}\right] \end{array} \begin{array}{l} \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

(iii) $2\% \text{ of } B = \frac{2}{100} \times B = 0.02 \times B$

$$= 0.02 \begin{array}{c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \left[\begin{array}{ccc} 5000 & 10,000 & 6000 \\ 20,000 & 10,000 & 10,000 \end{array}\right] \end{array} \begin{array}{l} \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

$$= \begin{array}{c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \left[\begin{array}{ccc} 100 & 200 & 120 \\ 400 & 200 & 200 \end{array}\right] \end{array} \begin{array}{l} \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

Thus, in October Ramkishan receives ₹ 100, ₹ 200 and ₹ 120 as profit in the sale of each variety of rice, respectively, and Grucharan Singh receives profit of ₹ 400, ₹ 200 and ₹ 200 in the sale of each variety of rice, respectively.

3.4.5 *Multiplication of matrices*

Suppose Meera and Nadeem are two friends. Meera wants to buy 2 pens and 5 story books, while Nadeem needs 8 pens and 10 story books. They both go to a shop to enquire about the rates which are quoted as follows:

Pen – ₹ 5 each, story book – ₹ 50 each.

How much money does each need to spend? Clearly, Meera needs ₹ $(5 \times 2 + 50 \times 5)$ that is ₹ 260, while Nadeem needs $(8 \times 5 + 50 \times 10)$ ₹, that is ₹ 540. In terms of matrix representation, we can write the above information as follows:

Requirements **Prices per piece (in Rupees)** **Money needed (in Rupees)**

$$\begin{bmatrix} 2 & 5 \\ 8 & 10 \end{bmatrix} \qquad \begin{bmatrix} 5 \\ 50 \end{bmatrix} \qquad \begin{bmatrix} 5 \times 2 + 5 \times 50 \\ 8 \times 5 + 10 \times 50 \end{bmatrix} = \begin{bmatrix} 260 \\ 540 \end{bmatrix}$$

Suppose that they enquire about the rates from another shop, quoted as follows:

pen – ₹ 4 each, story book – ₹ 40 each.

Now, the money required by Meera and Nadeem to make purchases will be respectively ₹ $(4 \times 2 + 40 \times 5)$ = ₹ 208 and ₹ $(8 \times 4 + 10 \times 40)$ = ₹ 432

Again, the above information can be represented as follows:

Requirements **Prices per piece (in Rupees)** **Money needed (in Rupees)**

$$\begin{bmatrix} 2 & 5 \\ 8 & 10 \end{bmatrix} \qquad \begin{bmatrix} 4 \\ 40 \end{bmatrix} \qquad \begin{bmatrix} 4\times 2+40\times 5 \\ 8\times 4+10\times 40 \end{bmatrix} = \begin{bmatrix} 208 \\ 432 \end{bmatrix}$$

Now, the information in both the cases can be combined and expressed in terms of matrices as follows:

Requirements **Prices per piece (in Rupees)** **Money needed (in Rupees)**

$$\begin{bmatrix} 2 & 5 \\ 8 & 10 \end{bmatrix} \qquad \begin{bmatrix} 5 & 4 \\ 50 & 40 \end{bmatrix} \qquad \begin{bmatrix} 5\times 2+5\times 50 & 4\times 2+40\times 5 \\ 8\times 5+10\times 50 & 8\times 4+10\times 40 \end{bmatrix}$$

$$= \begin{bmatrix} 260 & 208 \\ 540 & 432 \end{bmatrix}$$

The above is an example of multiplication of matrices. We observe that, for multiplication of two matrices A and B, the number of columns in A should be equal to the number of rows in B. Furthermore for getting the elements of the product matrix, we take rows of A and columns of B, multiply them element-wise and take the sum. Formally, we define multiplication of matrices as follows:

The *product* of two matrices A and B is *defined* if the number of columns of A is equal to the number of rows of B. Let A = $[a_{ij}]$ be an $m \times n$ matrix and B = $[b_{jk}]$ be an $n \times p$ matrix. Then the product of the matrices A and B is the matrix C of order $m \times p$. To get the $(i, k)^{\text{th}}$ element c_{ik} of the matrix C, we take the i^{th} row of A and k^{th} column of B, multiply them elementwise and take the sum of all these products. In other words, if A = $[a_{ij}]_{m \times n}$, B = $[b_{jk}]_{n \times p}$, then the i^{th} row of A is $[a_{i1}\ a_{i2}\ ...\ a_{in}]$ and the k^{th} column of

B is $\begin{bmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{nk} \end{bmatrix}$, then $c_{ik} = a_{i1} b_{1k} + a_{i2} b_{2k} + a_{i3} b_{3k} + ... + a_{in} b_{nk} = \sum_{j=1}^{n} a_{ij} b_{jk}$.

The matrix C = $[c_{ik}]_{m \times p}$ is the product of A and B.

For example, if $C = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix}$, then the product CD is defined and is given by $CD = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix}$. This is a 2×2 matrix in which each entry is the sum of the products across some row of C with the corresponding entries down some column of D. These four computations are

Entry in first row first column $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} (1)(2)+(-1)(-1)+(2)(5) & ? \\ ? & ? \end{bmatrix}$

Entry in first row second column $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} 13 & (1)(7)+(-1)(1)+2(-4) \\ ? & ? \end{bmatrix}$

Entry in second row first column $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} 13 & -2 \\ 0(2)+3(-1)+4(5) & ? \end{bmatrix}$

Entry in second row second column $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} 13 & -2 \\ 17 & 0(7)+3(1)+4(-4) \end{bmatrix}$

Thus $CD = \begin{bmatrix} 13 & -2 \\ 17 & -13 \end{bmatrix}$

Example 12 Find AB, if $A = \begin{bmatrix} 6 & 9 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 6 & 0 \\ 7 & 9 & 8 \end{bmatrix}$.

Solution The matrix A has 2 columns which is equal to the number of rows of B. Hence AB is defined. Now

$$AB = \begin{bmatrix} 6(2)+9(7) & 6(6)+9(9) & 6(0)+9(8) \\ 2(2)+3(7) & 2(6)+3(9) & 2(0)+3(8) \end{bmatrix}$$

$$= \begin{bmatrix} 12+63 & 36+81 & 0+72 \\ 4+21 & 12+27 & 0+24 \end{bmatrix} = \begin{bmatrix} 75 & 117 & 72 \\ 25 & 39 & 24 \end{bmatrix}$$

Remark If AB is defined, then BA need not be defined. In the above example, AB is defined but BA is not defined because B has 3 column while A has only 2 (and not 3) rows. If A, B are, respectively $m \times n$, $k \times l$ matrices, then both AB and BA are defined **if and only if** $n = k$ and $l = m$. In particular, if both A and B are square matrices of the same order, then both AB and BA are defined.

Non-commutativity of multiplication of matrices

Now, we shall see by an example that even if AB and BA are both defined, it is not necessary that AB = BA.

Example 13 If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$, then find AB, BA. Show that AB $\neq$ BA.

Solution Since A is a 2 × 3 matrix and B is 3 × 2 matrix. Hence AB and BA are both defined and are matrices of order 2 × 2 and 3 × 3, respectively. Note that

$$AB = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2-8+6 & 3-10+3 \\ -8+8+10 & -12+10+5 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$$

and $$BA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 2-12 & -4+6 & 6+15 \\ 4-20 & -8+10 & 12+25 \\ 2-4 & -4+2 & 6+5 \end{bmatrix} = \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}$$

Clearly AB $\neq$ BA

In the above example both AB and BA are of different order and so AB $\neq$ BA. But one may think that perhaps AB and BA could be the same if they were of the same order. But it is not so, here we give an example to show that even if AB and BA are of same order they may not be same.

Example 14 If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $AB = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

and $$BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$ Clearly AB $\neq$ BA.

Thus matrix multiplication is not commutative.

Note This does not mean that AB ≠ BA for every pair of matrices A, B for which AB and BA, are defined. For instance,

If $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$, then $AB = BA = \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix}$

Observe that multiplication of diagonal matrices of same order will be commutative.

Zero matrix as the product of two non zero matrices

We know that, for real numbers a, b if $ab = 0$, then either $a = 0$ or $b = 0$. This need not be true for matrices, we will observe this through an example.

Example 15 Find AB, if $A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$.

Solution We have $AB = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Thus, if the product of two matrices is a zero matrix, it is not necessary that one of the matrices is a zero matrix.

3.4.6 *Properties of multiplication of matrices*

The multiplication of matrices possesses the following properties, which we state without proof.

1. **The associative law** For any three matrices A, B and C. We have (AB) C = A (BC), whenever both sides of the equality are defined.
2. **The distributive law** For three matrices A, B and C.
   (i) A (B+C) = AB + AC
   (ii) (A+B) C = AC + BC, whenever both sides of equality are defined.
3. **The existence of multiplicative identity** For every square matrix A, there exist an identity matrix of same order such that IA = AI = A.

Now, we shall verify these properties by examples.

Example 16 If $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix}$, find A(BC), (AB)C and show that (AB)C = A(BC).

Solution We have $AB = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 1+0+1 & 3+2-4 \\ 2+0-3 & 6+0+12 \\ 3+0-2 & 9-2+8 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 18 \\ 1 & 15 \end{bmatrix}$

$$(AB)\ (C) = \begin{bmatrix} 2 & 1 \\ -1 & 18 \\ 1 & 15 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 2+2 & 4+0 & 6-2 & -8+1 \\ -1+36 & -2+0 & -3-36 & 4+18 \\ 1+30 & 2+0 & 3-30 & -4+15 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 & 4 & -7 \\ 35 & -2 & -39 & 22 \\ 31 & 2 & -27 & 11 \end{bmatrix}$$

Now $\quad BC = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1+6 & 2+0 & 3-6 & -4+3 \\ 0+4 & 0+0 & 0-4 & 0+2 \\ -1+8 & -2+0 & -3-8 & 4+4 \end{bmatrix}$

$$= \begin{bmatrix} 7 & 2 & -3 & -1 \\ 4 & 0 & -4 & 2 \\ 7 & -2 & -11 & 8 \end{bmatrix}$$

Therefore $\quad A(BC) = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 7 & 2 & -3 & -1 \\ 4 & 0 & -4 & 2 \\ 7 & -2 & -11 & 8 \end{bmatrix}$

$$= \begin{bmatrix} 7+4-7 & 2+0+2 & -3-4+11 & -1+2-8 \\ 14+0+21 & 4+0-6 & -6+0-33 & -2+0+24 \\ 21-4+14 & 6+0-4 & -9+4-22 & -3-2+16 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 & 4 & -7 \\ 35 & -2 & -39 & 22 \\ 31 & 2 & -27 & 11 \end{bmatrix}$$. Clearly, (AB) C = A (BC)

Example 17 If $A=\begin{bmatrix}0 & 6 & 7\\ -6 & 0 & 8\\ 7 & -8 & 0\end{bmatrix}, B=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 2\\ 1 & 2 & 0\end{bmatrix}, C=\begin{bmatrix}2\\ -2\\ 3\end{bmatrix}$

Calculate AC, BC and (A + B)C. Also, verify that (A + B)C = AC + BC

Solution Now, $A+B=\begin{bmatrix}0 & 7 & 8\\ -5 & 0 & 10\\ 8 & -6 & 0\end{bmatrix}$

So $$(A + B)\,C = \begin{bmatrix}0 & 7 & 8\\ -5 & 0 & 10\\ 8 & -6 & 0\end{bmatrix}\begin{bmatrix}2\\ -2\\ 3\end{bmatrix}=\begin{bmatrix}0-14+24\\ -10+0+30\\ 16+12+0\end{bmatrix}=\begin{bmatrix}10\\ 20\\ 28\end{bmatrix}$$

Further $$AC = \begin{bmatrix}0 & 6 & 7\\ -6 & 0 & 8\\ 7 & -8 & 0\end{bmatrix}\begin{bmatrix}2\\ -2\\ 3\end{bmatrix}=\begin{bmatrix}0-12+21\\ -12+0+24\\ 14+16+0\end{bmatrix}=\begin{bmatrix}9\\ 12\\ 30\end{bmatrix}$$

and $$BC = \begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 2\\ 1 & 2 & 0\end{bmatrix}\begin{bmatrix}2\\ -2\\ 3\end{bmatrix}=\begin{bmatrix}0-2+3\\ 2+0+6\\ 2-4+0\end{bmatrix}=\begin{bmatrix}1\\ 8\\ -2\end{bmatrix}$$

So $$AC + BC = \begin{bmatrix}9\\ 12\\ 30\end{bmatrix}+\begin{bmatrix}1\\ 8\\ -2\end{bmatrix}=\begin{bmatrix}10\\ 20\\ 28\end{bmatrix}$$

Clearly, (A + B) C = AC + BC

Example 18 If $A=\begin{bmatrix}1 & 2 & 3\\ 3 & -2 & 1\\ 4 & 2 & 1\end{bmatrix}$, then show that $A^3 - 23A - 40\,I = O$

Solution We have $A^2 = A.A = \begin{bmatrix}1 & 2 & 3\\ 3 & -2 & 1\\ 4 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ 3 & -2 & 1\\ 4 & 2 & 1\end{bmatrix}=\begin{bmatrix}19 & 4 & 8\\ 1 & 12 & 8\\ 14 & 6 & 15\end{bmatrix}$

So $$A^3 = A\,A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix} = \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix}$$

Now

$$A^3 - 23A - 40I = \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix} - 23 \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} - 40 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix} + \begin{bmatrix} -23 & -46 & -69 \\ -69 & 46 & -23 \\ -92 & -46 & -23 \end{bmatrix} + \begin{bmatrix} -40 & 0 & 0 \\ 0 & -40 & 0 \\ 0 & 0 & -40 \end{bmatrix}$$

$$= \begin{bmatrix} 63-23-40 & 46-46+0 & 69-69+0 \\ 69-69+0 & -6+46-40 & 23-23+0 \\ 92-92+0 & 46-46+0 & 63-23-40 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Example 19 In a legislative assembly election, a political group hired a public relations firm to promote its candidate in three ways: telephone, house calls, and letters. The cost per contact (in paise) is given in matrix A as

$$A = \begin{matrix} \textit{Cost per contact} \\ \begin{bmatrix} 40 \\ 100 \\ 50 \end{bmatrix} \end{matrix} \begin{matrix} \\ \text{Telephone} \\ \text{Housecall} \\ \text{Letter} \end{matrix}$$

The number of contacts of each type made in two cities X and Y is given by

$$B = \begin{matrix} \text{Telephone} & \text{Housecall} & \text{Letter} \\ \end{matrix} \atop \begin{bmatrix} 1000 & 500 & 5000 \\ 3000 & 1000 & 10{,}000 \end{bmatrix} \begin{matrix} \to X \\ \to Y \end{matrix}$$. Find the total amount spent by the group in the two cities X and Y.

Solution We have

$$BA = \begin{bmatrix} 40,000 + 50,000 + 250,000 \\ 120,000 + 100,000 + 500,000 \end{bmatrix} \begin{matrix} \to X \\ \to Y \end{matrix}$$

$$= \begin{bmatrix} 340,000 \\ 720,000 \end{bmatrix} \begin{matrix} \to X \\ \to Y \end{matrix}$$

So the total amount spent by the group in the two cities is 340,000 paise and 720,000 paise, i.e., ₹ 3400 and ₹ 7200, respectively.

EXERCISE 3.2

1. Let $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}, C = \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$

Find each of the following:

(i) A + B (ii) A – B (iii) 3A – C

(iv) AB (v) BA

2. Compute the following:

(i) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ (ii) $\begin{bmatrix} a^2+b^2 & b^2+c^2 \\ a^2+c^2 & a^2+b^2 \end{bmatrix} + \begin{bmatrix} 2ab & 2bc \\ -2ac & -2ab \end{bmatrix}$

(iii) $\begin{bmatrix} -1 & 4 & -6 \\ 8 & 5 & 16 \\ 2 & 8 & 5 \end{bmatrix} + \begin{bmatrix} 12 & 7 & 6 \\ 8 & 0 & 5 \\ 3 & 2 & 4 \end{bmatrix}$ (iv) $\begin{bmatrix} \cos^2 x & \sin^2 x \\ \sin^2 x & \cos^2 x \end{bmatrix} + \begin{bmatrix} \sin^2 x & \cos^2 x \\ \cos^2 x & \sin^2 x \end{bmatrix}$

3. Compute the indicated products.

(i) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ (ii) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & 4 \\ 3 & 0 & 5 \end{bmatrix}$ (v) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix}$

(vi) $\begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{bmatrix}$

4. If $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 3 & 2 \\ 1 & -2 & 3 \end{bmatrix}$, then compute (A+B) and (B – C). Also, verify that A + (B – C) = (A + B) – C.

5. If $A = \begin{bmatrix} \frac{2}{3} & 1 & \frac{5}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{4}{3} \\ \frac{7}{3} & 2 & \frac{2}{3} \end{bmatrix}$ and $B = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} & 1 \\ \frac{1}{5} & \frac{2}{5} & \frac{4}{5} \\ \frac{7}{5} & \frac{6}{5} & \frac{2}{5} \end{bmatrix}$, then compute 3A – 5B.

6. Simplify $\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

7. Find X and Y, if

(i) $X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

(ii) $2X + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$ and $3X + 2Y = \begin{bmatrix} 2 & -2 \\ -1 & 5 \end{bmatrix}$

8. Find X, if $Y = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ and $2X + Y = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$

9. Find x and y, if $2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$

10. Solve the equation for x, y, z and t, if $2\begin{bmatrix} x & z \\ y & t \end{bmatrix} + 3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = 3\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$

11. If $x\begin{bmatrix} 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$, find the values of x and y.

12. Given $3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$, find the values of x, y, z and w.

13. If $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$, show that $F(x)\, F(y) = F(x + y)$.

14. Show that

(i) $\begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \neq \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

15. Find $A^2 - 5A + 6I$, if $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

16. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$, prove that $A^3 - 6A^2 + 7A + 2I = 0$

17. If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find k so that $A^2 = kA - 2I$

18. If $A = \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ and I is the identity matrix of order 2, show that

$$I + A = (I - A) \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

19. A trust fund has ₹30,000 that must be invested in two different types of bonds. The first bond pays 5% interest per year, and the second bond pays 7% interest per year. Using matrix multiplication, determine how to divide ₹30,000 among the two types of bonds. If the trust fund must obtain an annual total interest of:

(a) ₹1800 (b) ₹2000

20. The bookshop of a particular school has 10 dozen chemistry books, 8 dozen physics books, 10 dozen economics books. Their selling prices are ₹ 80, ₹ 60 and ₹ 40 each respectively. Find the total amount the bookshop will receive from selling all the books using matrix algebra.

Assume X, Y, Z, W and P are matrices of order $2 \times n$, $3 \times k$, $2 \times p$, $n \times 3$ and $p \times k$, respectively. Choose the correct answer in Exercises 21 and 22.

21. The restriction on n, k and p so that PY + WY will be defined are:

(A) $k = 3, p = n$ (B) k is arbitrary, $p = 2$

(C) p is arbitrary, $k = 3$ (D) $k = 2, p = 3$

22. If $n = p$, then the order of the matrix 7X – 5Z is:

(A) $p \times 2$ (B) $2 \times n$ (C) $n \times 3$ (D) $p \times n$

3.5. Transpose of a Matrix

In this section, we shall learn about transpose of a matrix and special types of matrices such as symmetric and skew symmetric matrices.

Definition 3 If $A = [a_{ij}]$ be an $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of A is called the *transpose* of A. Transpose of the matrix A is denoted by A′ or (A^T). In other words, if $A = [a_{ij}]_{m \times n}$, then $A' = [a_{ji}]_{n \times m}$. For example,

$$\text{if } A = \begin{bmatrix} 3 & 5 \\ \sqrt{3} & 1 \\ 0 & \frac{-1}{5} \end{bmatrix}_{3 \times 2}, \text{ then } A' = \begin{bmatrix} 3 & \sqrt{3} & 0 \\ 5 & 1 & \frac{-1}{5} \end{bmatrix}_{2 \times 3}$$

3.5.1 Properties of transpose of the matrices

We now state the following properties of transpose of matrices without proof. These may be verified by taking suitable examples.

For any matrices A and B of suitable orders, we have

(i) $(A')' = A$, (ii) $(kA)' = kA'$ (where k is any constant)

(iii) $(A + B)' = A' + B'$ (iv) $(A B)' = B' A'$

Example 20 If $A = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$, verify that

(i) $(A')' = A$, (ii) $(A + B)' = A' + B'$,

(iii) $(kB)' = kB'$, where k is any constant.

Solution

(i) We have

$$A = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix} \Rightarrow A' = \begin{bmatrix} 3 & 4 \\ \sqrt{3} & 2 \\ 2 & 0 \end{bmatrix} \Rightarrow (A')' = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix} = A$$

Thus $(A')' = A$

(ii) We have

$$A = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix} \Rightarrow A + B = \begin{bmatrix} 5 & \sqrt{3}-1 & 4 \\ 5 & 4 & 4 \end{bmatrix}$$

Therefore $$(A + B)' = \begin{bmatrix} 5 & 5 \\ \sqrt{3}-1 & 4 \\ 4 & 4 \end{bmatrix}$$

Now $$A' = \begin{bmatrix} 3 & 4 \\ \sqrt{3} & 2 \\ 2 & 0 \end{bmatrix}, B' = \begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 2 & 4 \end{bmatrix},$$

So $$A' + B' = \begin{bmatrix} 5 & 5 \\ \sqrt{3}-1 & 4 \\ 4 & 4 \end{bmatrix}$$

Thus $(A + B)' = A' + B'$

(iii) We have

$$kB = k\begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2k & -k & 2k \\ k & 2k & 4k \end{bmatrix}$$

Then $$(kB)' = \begin{bmatrix} 2k & k \\ -k & 2k \\ 2k & 4k \end{bmatrix} = k\begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 2 & 4 \end{bmatrix} = kB'$$

Thus $(kB)' = kB'$

Example 21 If $A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & -6 \end{bmatrix}$, verify that $(AB)' = B'A'$.

Solution We have

$$A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & -6 \end{bmatrix}$$

then $$AB = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 3 & -6 \end{bmatrix} = \begin{bmatrix} -2 & -6 & 12 \\ 4 & 12 & -24 \\ 5 & 15 & -30 \end{bmatrix}$$

Now $$A' = [-2\ 4\ 5]\ ,\ B' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}$$

$$B'A' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix} \begin{bmatrix} -2 & 4 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{bmatrix} = (AB)'$$

Clearly $(AB)' = B'A'$

3.6 Symmetric and Skew Symmetric Matrices

Definition 4 A square matrix $A = [a_{ij}]$ is said to be *symmetric* if $A' = A$, that is, $[a_{ij}] = [a_{ji}]$ for all possible values of i and j.

For example $A = \begin{bmatrix} \sqrt{3} & 2 & 3 \\ 2 & -1.5 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ is a symmetric matrix as $A' = A$

Definition 5 A square matrix $A = [a_{ij}]$ is said to be *skew symmetric* matrix if $A' = -A$, that is $a_{ji} = -a_{ij}$ for all possible values of i and j. Now, if we put $i = j$, we have $a_{ii} = -a_{ii}$. Therefore $2a_{ii} = 0$ or $a_{ii} = 0$ for all i's.

This means that all the diagonal elements of a skew symmetric matrix are zero.

For example, the matrix $B = \begin{bmatrix} 0 & e & f \\ -e & 0 & g \\ -f & -g & 0 \end{bmatrix}$ is a skew symmetric matrix as B′= –B

Now, we are going to prove some results of symmetric and skew-symmetric matrices.

Theorem 1 For any square matrix A with real number entries, A + A′ is a symmetric matrix and A – A′ is a skew symmetric matrix.

Proof Let B = A + A′, then

$$
\begin{aligned}
B' &= (A + A')' \\
&= A' + (A')' \text{ (as } (A + B)' = A' + B') \\
&= A' + A \text{ (as } (A')' = A) \\
&= A + A' \text{ (as } A + B = B + A) \\
&= B
\end{aligned}
$$

Therefore $B = A + A'$ is a symmetric matrix

Now let $C = A - A'$

$$
\begin{aligned}
C' &= (A - A')' = A' - (A')' \quad \text{(Why?)} \\
&= A' - A \quad \text{(Why?)} \\
&= -(A - A') = -C
\end{aligned}
$$

Therefore $C = A - A'$ is a skew symmetric matrix.

Theorem 2 Any square matrix can be expressed as the sum of a symmetric and a skew symmetric matrix.

Proof Let A be a square matrix, then we can write

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

From the Theorem 1, we know that (A + A′) is a symmetric matrix and (A – A′) is a skew symmetric matrix. Since for any matrix A, $(kA)' = kA'$, it follows that $\frac{1}{2}(A + A')$

is symmetric matrix and $\frac{1}{2}(A - A')$ is skew symmetric matrix. Thus, any square matrix can be expressed as the sum of a symmetric and a skew symmetric matrix.

Example 22 Express the matrix $B = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrix.

Solution Here

$$B' = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{bmatrix}$$

Let $$P = \frac{1}{2}(B + B') = \frac{1}{2}\begin{bmatrix} 4 & -3 & -3 \\ -3 & 6 & 2 \\ -3 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix},$$

Now $$P' = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix} = P$$

Thus $P = \frac{1}{2}(B + B')$ is a symmetric matrix.

Also, let $$Q = \frac{1}{2}(B - B') = \frac{1}{2}\begin{bmatrix} 0 & -1 & -5 \\ 1 & 0 & 6 \\ 5 & -6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix}$$

Then $$Q' = \begin{bmatrix} 0 & \frac{1}{2} & \frac{5}{3} \\ \frac{-1}{2} & 0 & -3 \\ \frac{-5}{2} & 3 & 0 \end{bmatrix} = -Q$$

Thus $$Q = \frac{1}{2}(B - B') \text{ is a skew symmetric matrix.}$$

Now $$P + Q = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix} + \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = B$$

Thus, B is represented as the sum of a symmetric and a skew symmetric matrix.

EXERCISE 3.3

1. Find the transpose of each of the following matrices:

(i) $\begin{bmatrix} 5 \\ \frac{1}{2} \\ -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} -1 & 5 & 6 \\ \sqrt{3} & 5 & 6 \\ 2 & 3 & -1 \end{bmatrix}$

2. If $A = \begin{bmatrix} -1 & 2 & 3 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 1 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$, then verify that

(i) $(A + B)' = A' + B'$, (ii) $(A - B)' = A' - B'$

3. If $A' = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, then verify that

(i) $(A + B)' = A' + B'$ (ii) $(A - B)' = A' - B'$

4. If $A' = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$, then find $(A + 2B)'$

5. For the matrices A and B, verify that (AB)′ = B′A′, where

(i) $A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 5 & 7 \end{bmatrix}$

6. If (i) $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then verify that $A'A = I$

(ii) If $A = \begin{bmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{bmatrix}$, then verify that $A'A = I$

7. (i) Show that the matrix $A = \begin{bmatrix} 1 & -1 & 5 \\ -1 & 2 & 1 \\ 5 & 1 & 3 \end{bmatrix}$ is a symmetric matrix.

(ii) Show that the matrix $A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ is a skew symmetric matrix.

8. For the matrix $A = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$, verify that

(i) (A + A′) is a symmetric matrix

(ii) (A – A′) is a skew symmetric matrix

9. Find $\frac{1}{2}(A + A')$ and $\frac{1}{2}(A - A')$, when $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$

10. Express the following matrices as the sum of a symmetric and a skew symmetric matrix:

(i) $\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix}$

Choose the correct answer in the Exercises 11 and 12.

11. If A, B are symmetric matrices of same order, then AB – BA is a

(A) Skew symmetric matrix (B) Symmetric matrix

(C) Zero matrix (D) Identity matrix

12. If $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, and $A + A' = I$, then the value of α is

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$

(C) π (D) $\frac{3\pi}{2}$

3.7 Invertible Matrices

Definition 6 If A is a square matrix of order *m*, and if there exists another square matrix B of the same order *m*, such that AB = BA = I, then B is called the *inverse* matrix of A and it is denoted by A^{-1}. In that case A is said to be invertible.

For example, let $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ be two matrices.

Now
$$AB = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 4-3 & -6+6 \\ 2-2 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Also $BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$. Thus B is the inverse of A, in other words $B = A^{-1}$ and A is inverse of B, i.e., $A = B^{-1}$

Note

1. A rectangular matrix does not possess inverse matrix, since for products BA and AB to be defined and to be equal, it is necessary that matrices A and B should be square matrices of the same order.
2. If B is the inverse of A, then A is also the inverse of B.

Theorem 3 (Uniqueness of inverse) Inverse of a square matrix, if it exists, is unique.

Proof Let A = $[a_{ij}]$ be a square matrix of order m. If possible, let B and C be two inverses of A. We shall show that B = C.

Since B is the inverse of A

$$AB = BA = I \quad \text{... (1)}$$

Since C is also the inverse of A

$$AC = CA = I \quad \text{... (2)}$$

Thus $\quad B = BI = B (AC) = (BA) C = IC = C$

Theorem 4 If A and B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1} A^{-1}$.

Proof From the definition of inverse of a matrix, we have

$$(AB)\ (AB)^{-1} = 1$$

or $\quad A^{-1}\ (AB)\ (AB)^{-1} = A^{-1}I \quad$ (Pre multiplying both sides by A^{-1})

or $\quad (A^{-1}A)\ B\ (AB)^{-1} = A^{-1} \quad$ (Since $A^{-1}\ I = A^{-1}$)

or $\quad IB\ (AB)^{-1} = A^{-1}$

or $\quad B\ (AB)^{-1} = A^{-1}$

or $\quad B^{-1}\ B\ (AB)^{-1} = B^{-1}\ A^{-1}$

or $\quad I\ (AB)^{-1} = B^{-1}\ A^{-1}$

Hence $\quad (AB)^{-1} = B^{-1}\ A^{-1}$

EXERCISE 3.4

1. Matrices A and B will be inverse of each other only if

(A) AB = BA (B) AB = BA = 0

(C) AB = 0, BA = I (D) AB = BA = I

Miscellaneous Examples

Example 23 If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then prove that $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$, $n \in \mathbf{N}$.

Solution We shall prove the result by using principle of mathematical induction.

We have $P(n)$: If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$, $n \in \mathbf{N}$

$$P(1):\ A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},\ \text{so}\ A^1 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Therefore, the result is true for $n = 1$.

Let the result be true for $n = k$. So

$$P(k):\ A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},\ \text{then}\ A^k = \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$$

Now, we prove that the result holds for $n = k + 1$

Now
$$A^{k+1} = A \cdot A^k = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos k\theta - \sin\theta\sin k\theta & \cos\theta\sin k\theta + \sin\theta\cos k\theta \\ -\sin\theta\cos k\theta + \cos\theta\sin k\theta & -\sin\theta\sin k\theta + \cos\theta\cos k\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta + k\theta) & \sin(\theta + k\theta) \\ -\sin(\theta + k\theta) & \cos(\theta + k\theta) \end{bmatrix} = \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$$

Therefore, the result is true for $n = k + 1$. Thus by principle of mathematical induction, we have $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$, holds for all natural numbers.

Example 24 If A and B are symmetric matrices of the same order, then show that AB is symmetric if and only if A and B commute, that is AB = BA.

Solution Since A and B are both symmetric matrices, therefore $A' = A$ and $B' = B$.

Let AB be symmetric, then (AB)′ = AB

But (AB)′ = B′A′= BA (Why?)

Therefore BA = AB

Conversely, if AB = BA, then we shall show that AB is symmetric.

Now (AB)′ = B′A′

= B A (as A and B are symmetric)

= AB

Hence AB is symmetric.

Example 25 Let $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}, C = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$. Find a matrix D such that CD – AB = O.

Solution Since A, B, C are all square matrices of order 2, and CD – AB is well defined, D must be a square matrix of order 2.

Let $D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then CD – AB = 0 gives

$$\begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix} = O$$

or $$\begin{bmatrix} 2a+5c & 2b+5d \\ 3a+8c & 3b+8d \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 43 & 22 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

or $$\begin{bmatrix} 2a+5c-3 & 2b+5d \\ 3a+8c-43 & 3b+8d-22 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

By equality of matrices, we get

$$2a + 5c - 3 = 0 \quad \text{... (1)}$$

$$3a + 8c - 43 = 0 \quad \text{... (2)}$$

$$2b + 5d = 0 \quad \text{... (3)}$$

and $$3b + 8d - 22 = 0 \quad \text{... (4)}$$

Solving (1) and (2), we get $a = -191$, $c = 77$. Solving (3) and (4), we get $b = -110$, $d = 44$.

Therefore $$D = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -191 & -110 \\ 77 & 44 \end{bmatrix}$$

Miscellaneous Exercise on Chapter 3

1. If A and B are symmetric matrices, prove that AB – BA is a skew symmetric matrix.

2. Show that the matrix B′AB is symmetric or skew symmetric according as A is symmetric or skew symmetric.

3. Find the values of x, y, z if the matrix $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ satisfy the equation $A'A = I$.

4. For what values of x : $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = O$?

5. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = 0$.

6. Find x, if $\begin{bmatrix} x & -5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = O$

7. A manufacturer produces three products x, y, z which he sells in two markets. Annual sales are indicated below:

Market	**Products**		
I	10,000	2,000	18,000
II	6,000	20,000	8,000

(a) If unit sale prices of x, y and z are ₹ 2.50, ₹ 1.50 and ₹ 1.00, respectively, find the total revenue in each market with the help of matrix algebra.

(b) If the unit costs of the above three commodities are ₹ 2.00, ₹ 1.00 and 50 paise respectively. Find the gross profit.

8. Find the matrix X so that $X \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$

Choose the correct answer in the following questions:

9. If $A = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ is such that $A^2 = I$, then

(A) $1 + \alpha^2 + \beta\gamma = 0$ (B) $1 - \alpha^2 + \beta\gamma = 0$

(C) $1 - \alpha^2 - \beta\gamma = 0$ (D) $1 + \alpha^2 - \beta\gamma = 0$

10. If the matrix A is both symmetric and skew symmetric, then

(A) A is a diagonal matrix (B) A is a zero matrix

(C) A is a square matrix (D) None of these

11. If A is square matrix such that $A^2 = A$, then $(I + A)^3 - 7A$ is equal to

(A) A (B) I – A (C) I (D) 3A

Summary

- A matrix is an ordered rectangular array of numbers or functions.
- A matrix having m rows and n columns is called a matrix of order $m \times n$.
- $[a_{ij}]_{m \times 1}$ is a column matrix.
- $[a_{ij}]_{1 \times n}$ is a row matrix.
- An $m \times n$ matrix is a square matrix if $m = n$.
- $A = [a_{ij}]_{m \times m}$ is a diagonal matrix if $a_{ij} = 0$, when $i \neq j$.
- $A = [a_{ij}]_{n \times n}$ is a scalar matrix if $a_{ij} = 0$, when $i \neq j$, $a_{ij} = k$, (k is some constant), when $i = j$.
- $A = [a_{ij}]_{n \times n}$ is an identity matrix, if $a_{ij} = 1$, when $i = j$, $a_{ij} = 0$, when $i \neq j$.
- A zero matrix has all its elements as zero.
- $A = [a_{ij}] = [b_{ij}] = B$ if (i) A and B are of same order, (ii) $a_{ij} = b_{ij}$ for all possible values of i and j.

- $k\text{A} = k[a_{ij}]_{m \times n} = [k(a_{ij})]_{m \times n}$
- $-\text{A} = (-1)\text{A}$
- $\text{A} - \text{B} = \text{A} + (-1)\ \text{B}$
- $\text{A} + \text{B} = \text{B} + \text{A}$
- $(\text{A} + \text{B}) + \text{C} = \text{A} + (\text{B} + \text{C})$, where A, B and C are of same order.
- $k(\text{A} + \text{B}) = k\text{A} + k\text{B}$, where A and B are of same order, k is constant.
- $(k + l)\ \text{A} = k\text{A} + l\text{A}$, where k and l are constant.
- If $\text{A} = [a_{ij}]_{m \times n}$ and $\text{B} = [b_{jk}]_{n \times p}$, then $\text{AB} = \text{C} = [c_{ik}]_{m \times p}$, where $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$
- (i) $\text{A(BC)} = \text{(AB)C}$, (ii) $\text{A(B + C)} = \text{AB} + \text{AC}$, (iii) $\text{(A + B)C} = \text{AC} + \text{BC}$
- If $\text{A} = [a_{ij}]_{m \times n}$, then A' or $\text{A}^{\text{T}} = [a_{ji}]_{n \times m}$
- (i) $(\text{A}')' = \text{A}$, (ii) $(k\text{A})' = k\text{A}'$, (iii) $(\text{A} + \text{B})' = \text{A}' + \text{B}'$, (iv) $(\text{AB})' = \text{B}'\text{A}'$
- A is a symmetric matrix if $\text{A}' = \text{A}$.
- A is a skew symmetric matrix if $\text{A}' = -\text{A}$.
- Any square matrix can be represented as the sum of a symmetric and a skew symmetric matrix.
- If A and B are two square matrices such that AB = BA = I, then B is the inverse matrix of A and is denoted by A^{-1} and A is the inverse of B.
- Inverse of a square matrix, if it exists, is unique.

NOTES

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Chapter 4

DETERMINANTS

❖ *All Mathematical truths are relative and conditional. — C.P. STEINMETZ* ❖

4.1 Introduction

In the previous chapter, we have studied about matrices and algebra of matrices. We have also learnt that a system of algebraic equations can be expressed in the form of matrices. This means, a system of linear equations like

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

can be represented as $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. Now, this system of equations has a unique solution or not, is determined by the number $a_1 b_2 - a_2 b_1$. (Recall that if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ or, $a_1 b_2 - a_2 b_1 \neq 0$, then the system of linear equations has a unique solution). The number $a_1 b_2 - a_2 b_1$ which determines uniqueness of solution is associated with the matrix $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ and is called the determinant of A or det A. Determinants have wide applications in Engineering, Science, Economics, Social Science, etc.

P.S. Laplace (1749-1827)

In this chapter, we shall study determinants up to order three only with real entries. Also, we will study various properties of determinants, minors, cofactors and applications of determinants in finding the area of a triangle, adjoint and inverse of a square matrix, consistency and inconsistency of system of linear equations and solution of linear equations in two or three variables using inverse of a matrix.

4.2 Determinant

To every square matrix $A = [a_{ij}]$ of order n, we can associate a number (real or complex) called determinant of the square matrix A, where $a_{ij} = (i, j)^{th}$ element of A.

This may be thought of as a function which associates each square matrix with a unique number (real or complex). If M is the set of square matrices, K is the set of numbers (real or complex) and $f : M \to K$ is defined by $f(A) = k$, where $A \in M$ and $k \in K$, then $f(A)$ is called the determinant of A. It is also denoted by | A | or det A or Δ.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then determinant of A is written as $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det(A)$

Remarks

(i) For matrix A, | A | is read as determinant of A and not modulus of A.

(ii) Only square matrices have determinants.

4.2.1 *Determinant of a matrix of order one*

Let A = [*a*] be the matrix of order 1, then determinant of A is defined to be equal to *a*

4.2.2 *Determinant of a matrix of order two*

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a matrix of order 2 × 2,

then the determinant of A is defined as:

$$\det(A) = |A| = \Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Example 1 Evaluate $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}$.

Solution We have $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = 2(2) - 4(-1) = 4 + 4 = 8$.

Example 2 Evaluate $\begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix}$

Solution We have

$$\begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix} = x(x) - (x+1)(x-1) = x^2 - (x^2 - 1) = x^2 - x^2 + 1 = 1$$

4.2.3 *Determinant of a matrix of order 3 × 3*

Determinant of a matrix of order three can be determined by expressing it in terms of second order determinants. This is known as expansion of a determinant along a row (or a column). There are six ways of expanding a determinant of order

3 corresponding to each of three rows (R_1, R_2 and R_3) and three columns (C_1, C_2 and C_3) giving the same value as shown below.

Consider the determinant of square matrix A = $[a_{ij}]_{3 \times 3}$

i.e., $$|A| = \begin{vmatrix} \boldsymbol{a_{11}} & \boldsymbol{a_{12}} & \boldsymbol{a_{13}} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expansion along first Row (R_1)

Step 1 Multiply first element a_{11} of R_1 by $(-1)^{(1+1)}$ $[(-1)^{\text{sum of suffixes in } a_{11}}]$ and with the second order determinant obtained by deleting the elements of first row (R_1) and first column (C_1) of | A | as a_{11} lies in R_1 and C_1,

i.e., $$(-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Step 2 Multiply 2nd element a_{12} of R_1 by $(-1)^{1+2}$ $[(-1)^{\text{sum of suffixes in } a_{12}}]$ and the second order determinant obtained by deleting elements of first row (R_1) and 2nd column (C_2) of | A | as a_{12} lies in R_1 and C_2,

i.e., $$(-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

Step 3 Multiply third element a_{13} of R_1 by $(-1)^{1+3}$ $[(-1)^{\text{sum of suffixes in } a_{13}}]$ and the second order determinant obtained by deleting elements of first row (R_1) and third column (C_3) of | A | as a_{13} lies in R_1 and C_3,

i.e., $$(-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Step 4 Now the expansion of determinant of A, that is, | A | written as sum of all three terms obtained in steps 1, 2 and 3 above is given by

$$\det A = |A| = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+ (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

or $$|A| = a_{11}(a_{22} a_{33} - a_{32} a_{23}) - a_{12}(a_{21} a_{33} - a_{31} a_{23}) + a_{13}(a_{21} a_{32} - a_{31} a_{22})$$

$$= a_{11}\, a_{22}\, a_{33} - a_{11}\, a_{32}\, a_{23} - a_{12}\, a_{21}\, a_{33} + a_{12}\, a_{31}\, a_{23} + a_{13}\, a_{21}\, a_{32} - a_{13}\, a_{31}\, a_{22} \quad \ldots (1)$$

Note We shall apply all four steps together.

Expansion along second row (R_2)

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ \boldsymbol{a_{21}} & \boldsymbol{a_{22}} & \boldsymbol{a_{23}} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding along R_2, we get

$$|A| = (-1)^{2+1} a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{2+2} a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{2+3} a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= -a_{21}\,(a_{12}\, a_{33} - a_{32}\, a_{13}) + a_{22}\,(a_{11}\, a_{33} - a_{31}\, a_{13}) - a_{23}\,(a_{11}\, a_{32} - a_{31}\, a_{12})$$

$$|A| = -a_{21}\, a_{12}\, a_{33} + a_{21}\, a_{32}\, a_{13} + a_{22}\, a_{11}\, a_{33} - a_{22}\, a_{31}\, a_{13} - a_{23}\, a_{11}\, a_{32} + a_{23}\, a_{31}\, a_{12}$$

$$= a_{11}\, a_{22}\, a_{33} - a_{11}\, a_{23}\, a_{32} - a_{12}\, a_{21}\, a_{33} + a_{12}\, a_{23}\, a_{31} + a_{13}\, a_{21}\, a_{32} - a_{13}\, a_{31}\, a_{22} \quad \ldots (2)$$

Expansion along first Column (C_1)

$$|A| = \begin{vmatrix} \boldsymbol{a_{11}} & a_{12} & a_{13} \\ \boldsymbol{a_{21}} & a_{22} & a_{23} \\ \boldsymbol{a_{31}} & a_{32} & a_{33} \end{vmatrix}$$

By expanding along C_1, we get

$$|A| = a_{11}\,(-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21}\,(-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\,(-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11}\,(a_{22}\, a_{33} - a_{23}\, a_{32}) - a_{21}\,(a_{12}\, a_{33} - a_{13}\, a_{32}) + a_{31}\,(a_{12}\, a_{23} - a_{13}\, a_{22})$$

$$|A| = a_{11}\, a_{22}\, a_{33} - a_{11}\, a_{23}\, a_{32} - a_{21}\, a_{12}\, a_{33} + a_{21}\, a_{13}\, a_{32} + a_{31}\, a_{12}\, a_{23} - a_{31}\, a_{13}\, a_{22}$$

$$= a_{11}\, a_{22}\, a_{33} - a_{11}\, a_{23}\, a_{32} - a_{12}\, a_{21}\, a_{33} + a_{12}\, a_{23}\, a_{31} + a_{13}\, a_{21}\, a_{32} - a_{13}\, a_{31}\, a_{22} \quad \text{... (3)}$$

Clearly, values of |A| in (1), (2) and (3) are equal. It is left as an exercise to the reader to verify that the values of |A| by expanding along R_3, C_2 and C_3 are equal to the value of |A| obtained in (1), (2) or (3).

Hence, expanding a determinant along any row or column gives same value.

Remarks

(i) For easier calculations, we shall expand the determinant along that row or column which contains maximum number of zeros.

(ii) While expanding, instead of multiplying by $(-1)^{i+j}$, we can multiply by +1 or –1 according as $(i + j)$ is even or odd.

(iii) Let $A = \begin{bmatrix} 2 & 2 \\ 4 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$. Then, it is easy to verify that A = 2B. Also $|A| = 0 - 8 = -8$ and $|B| = 0 - 2 = -2$.

Observe that, $|A| = 4(-2) = 2^2|B|$ or $|A| = 2^n|B|$, where $n = 2$ is the order of square matrices A and B.

In general, if $A = kB$ where A and B are square matrices of order n, then $|A| = k^n |B|$, where $n = 1, 2, 3$

Example 3 Evaluate the determinant $\Delta = \begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix}$.

Solution Note that in the third column, two entries are zero. So expanding along third column (C_3), we get

$$\Delta = 4\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}$$

$$= 4\,(-1 - 12) - 0 + 0 = -52$$

Example 4 Evaluate $\Delta = \begin{vmatrix} 0 & \sin\alpha & -\cos\alpha \\ -\sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix}$.

Solution Expanding along R_1, we get

$$\Delta = 0\begin{vmatrix} 0 & \sin\beta \\ -\sin\beta & 0 \end{vmatrix} - \sin\alpha\begin{vmatrix} -\sin\alpha & \sin\beta \\ \cos\alpha & 0 \end{vmatrix} - \cos\alpha\begin{vmatrix} -\sin\alpha & 0 \\ \cos\alpha & -\sin\beta \end{vmatrix}$$

$$= 0 - \sin\alpha\,(0 - \sin\beta\cos\alpha) - \cos\alpha\,(\sin\alpha\sin\beta - 0)$$

$$= \sin\alpha\sin\beta\cos\alpha - \cos\alpha\sin\alpha\sin\beta = 0$$

Example 5 Find values of x for which $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$.

Solution We have $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$

i.e. $3 - x^2 = 3 - 8$

i.e. $x^2 = 8$

Hence $x = \pm 2\sqrt{2}$

EXERCISE 4.1

Evaluate the determinants in Exercises 1 and 2.

1. $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$

2. (i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$ (ii) $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

3. If $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, then show that $|2A| = 4|A|$

4. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, then show that $|3A| = 27|A|$

5. Evaluate the determinants

(i) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$ (ii) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

(iii) $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$ (iv) $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

6. If $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}$, find $|A|$

7. Find values of x, if

(i) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$ (ii) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

8. If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$, then x is equal to

(A) 6 (B) ± 6 (C) – 6 (D) 0

4.3 Area of a Triangle

In earlier classes, we have studied that the area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3), is given by the expression $\frac{1}{2}[x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2)]$. Now this expression can be written in the form of a determinant as

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \text{... (1)}$$

Remarks

(i) Since area is a positive quantity, we always take the absolute value of the determinant in (1).

(ii) If area is given, use both positive and negative values of the determinant for calculation.

(iii) The area of the triangle formed by three collinear points is zero.

Example 6 Find the area of the triangle whose vertices are (3, 8), (– 4, 2) and (5, 1).

Solution The area of triangle is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} 3 & 8 & 1 \\ -4 & 2 & 1 \\ 5 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[3(2-1)-8(-4-5)+1(-4-10)\right]$$

$$= \frac{1}{2}(3+72-14) = \frac{61}{2}$$

Example 7 Find the equation of the line joining A(1, 3) and B (0, 0) using determinants and find k if D(k, 0) is a point such that area of triangle ABD is 3sq units.

Solution Let P (x, y) be any point on AB. Then, area of triangle ABP is zero (Why?). So

$$\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 3 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

This gives $\frac{1}{2}(y-3x) = 0$ or $y = 3x$,

which is the equation of required line AB.

Also, since the area of the triangle ABD is 3 sq. units, we have

$$\frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ k & 0 & 1 \end{vmatrix} = \pm 3$$

This gives, $\frac{-3k}{2} = \pm 3$, i.e., $k = \mp 2$.

EXERCISE 4.2

1. Find area of the triangle with vertices at the point given in each of the following :
   (i) (1, 0), (6, 0), (4, 3) (ii) (2, 7), (1, 1), (10, 8)
   (iii) (–2, –3), (3, 2), (–1, –8)
2. Show that points
   A (a, $b + c$), B (b, $c + a$), C (c, $a + b$) are collinear.
3. Find values of k if area of triangle is 4 sq. units and vertices are
   (i) (k, 0), (4, 0), (0, 2) (ii) (–2, 0), (0, 4), (0, k)
4. (i) Find equation of line joining (1, 2) and (3, 6) using determinants.
   (ii) Find equation of line joining (3, 1) and (9, 3) using determinants.
5. If area of triangle is 35 sq units with vertices (2, – 6), (5, 4) and (k, 4). Then k is
   (A) 12 (B) –2 (C) –12, –2 (D) 12, –2

4.4 Minors and Cofactors

In this section, we will learn to write the expansion of a determinant in compact form using minors and cofactors.

Definition 1 Minor of an element a_{ij} of a determinant is the determinant obtained by deleting its ith row and jth column in which element a_{ij} lies. Minor of an element a_{ij} is denoted by M_{ij}.

Remark Minor of an element of a determinant of order $n(n \geq 2)$ is a determinant of order $n - 1$.

Example 8 Find the minor of element 6 in the determinant $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

Solution Since 6 lies in the second row and third column, its minor M_{23} is given by

$$M_{23} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = 8 - 14 = -6 \text{ (obtained by deleting } R_2 \text{ and } C_3 \text{ in } \Delta).$$

Definition 2 Cofactor of an element a_{ij}, denoted by A_{ij} is defined by

$$A_{ij} = (-1)^{i+j} M_{ij}, \text{ where } M_{ij} \text{ is minor of } a_{ij}.$$

Example 9 Find minors and cofactors of all the elements of the determinant $\begin{vmatrix} 1 & -2 \\ 4 & 3 \end{vmatrix}$

Solution Minor of the element a_{ij} is M_{ij}

Here $a_{11} = 1$. So M_{11} = Minor of $a_{11} = 3$

M_{12} = Minor of the element $a_{12} = 4$

M_{21} = Minor of the element $a_{21} = -2$

M_{22} = Minor of the element $a_{22} = 1$

Now, cofactor of a_{ij} is A_{ij}. So

$A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (3) = 3$

$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (4) = -4$

$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (-2) = 2$

$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (1) = 1$

Example 10 Find minors and cofactors of the elements a_{11}, a_{21} in the determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Solution By definition of minors and cofactors, we have

$$\text{Minor of } a_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}\, a_{33} - a_{23}\, a_{32}$$

$$\text{Cofactor of } a_{11} = A_{11} = (-1)^{1+1} \; M_{11} = a_{22}\, a_{33} - a_{23}\, a_{32}$$

$$\text{Minor of } a_{21} = M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12}\, a_{33} - a_{13}\, a_{32}$$

$$\text{Cofactor of } a_{21} = A_{21} = (-1)^{2+1} \; M_{21} = (-1)\,(a_{12}\, a_{33} - a_{13}\, a_{32}) = -\, a_{12}\, a_{33} + a_{13}\, a_{32}$$

Remark Expanding the determinant Δ, in Example 21, along R_1, we have

$$\Delta = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} \; a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$= a_{11}\, A_{11} + a_{12}\, A_{12} + a_{13}\, A_{13}$, where A_{ij} is cofactor of a_{ij}

= sum of product of elements of R_1 with their corresponding cofactors

Similarly, Δ can be calculated by other five ways of expansion that is along R_2, R_3, C_1, C_2 and C_3.

Hence Δ = sum of the product of elements of any row (or column) with their corresponding cofactors.

Note If elements of a row (or column) are multiplied with cofactors of any other row (or column), then their sum is zero. For example,

$$\Delta = a_{11}\, A_{21} + a_{12}\, A_{22} + a_{13}\, A_{23}$$

$$= a_{11} \; (-1)^{1+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{12} \; (-1)^{1+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \; (-1)^{1+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 \text{ (since } R_1 \text{ and } R_2 \text{ are identical)}$$

Similarly, we can try for other rows and columns.

Example 11 Find minors and cofactors of the elements of the determinant

$$\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$$ and verify that $a_{11} A_{31} + a_{12} A_{32} + a_{13} A_{33} = 0$

Solution We have $M_{11} = \begin{vmatrix} 0 & 4 \\ 5 & -7 \end{vmatrix} = 0 - 20 = -20$; $A_{11} = (-1)^{1+1}(-20) = -20$

$$M_{12} = \begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix} = -42 - 4 = -46; \qquad A_{12} = (-1)^{1+2}(-46) = 46$$

$$M_{13} = \begin{vmatrix} 6 & 0 \\ 1 & 5 \end{vmatrix} = 30 - 0 = 30; \qquad A_{13} = (-1)^{1+3}(30) = 30$$

$$M_{21} = \begin{vmatrix} -3 & 5 \\ 5 & -7 \end{vmatrix} = 21 - 25 = -4; \qquad A_{21} = (-1)^{2+1}(-4) = 4$$

$$M_{22} = \begin{vmatrix} 2 & 5 \\ 1 & -7 \end{vmatrix} = -14 - 5 = -19; \qquad A_{22} = (-1)^{2+2}(-19) = -19$$

$$M_{23} = \begin{vmatrix} 2 & -3 \\ 1 & 5 \end{vmatrix} = 10 + 3 = 13; \qquad A_{23} = (-1)^{2+3}(13) = -13$$

$$M_{31} = \begin{vmatrix} -3 & 5 \\ 0 & 4 \end{vmatrix} = -12 - 0 = -12; \qquad A_{31} = (-1)^{3+1}(-12) = -12$$

$$M_{32} = \begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix} = 8 - 30 = -22; \qquad A_{32} = (-1)^{3+2}(-22) = 22$$

and $M_{33} = \begin{vmatrix} 2 & -3 \\ 6 & 0 \end{vmatrix} = 0 + 18 = 18$; $\qquad A_{33} = (-1)^{3+3}(18) = 18$

Now $a_{11} = 2, a_{12} = -3, a_{13} = 5$; $A_{31} = -12, A_{32} = 22, A_{33} = 18$

So $a_{11} A_{31} + a_{12} A_{32} + a_{13} A_{33}$

$= 2(-12) + (-3)(22) + 5(18) = -24 - 66 + 90 = 0$

EXERCISE 4.3

Write Minors and Cofactors of the elements of following determinants:

1. (i) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$ (ii) $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

2. (i) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

3. Using Cofactors of elements of second row, evaluate $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$.

4. Using Cofactors of elements of third column, evaluate $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$.

5. If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and A_{ij} is Cofactors of a_{ij}, then value of Δ is given by

(A) $a_{11} A_{31} + a_{12} A_{32} + a_{13} A_{33}$ (B) $a_{11} A_{11} + a_{12} A_{21} + a_{13} A_{31}$

(C) $a_{21} A_{11} + a_{22} A_{12} + a_{23} A_{13}$ (D) $a_{11} A_{11} + a_{21} A_{21} + a_{31} A_{31}$

4.5 Adjoint and Inverse of a Matrix

In the previous chapter, we have studied inverse of a matrix. In this section, we shall discuss the condition for existence of inverse of a matrix.

To find inverse of a matrix A, i.e., A^{-1} we shall first define adjoint of a matrix.

4.5.1 *Adjoint of a matrix*

Definition 3 The adjoint of a square matrix $A = [a_{ij}]_{n \times n}$ is defined as the transpose of the matrix $[A_{ij}]_{n \times n}$, where A_{ij} is the cofactor of the element a_{ij}. Adjoint of the matrix A is denoted by *adj* A.

Let $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then $adj\, \text{A} = \text{Transpose of} \begin{bmatrix} \text{A}_{11} & \text{A}_{12} & \text{A}_{13} \\ \text{A}_{21} & \text{A}_{22} & \text{A}_{23} \\ \text{A}_{31} & \text{A}_{32} & \text{A}_{33} \end{bmatrix} = \begin{bmatrix} \text{A}_{11} & \text{A}_{21} & \text{A}_{31} \\ \text{A}_{12} & \text{A}_{22} & \text{A}_{32} \\ \text{A}_{13} & \text{A}_{23} & \text{A}_{33} \end{bmatrix}$

Example 12 Find *adj* A for $\text{A} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

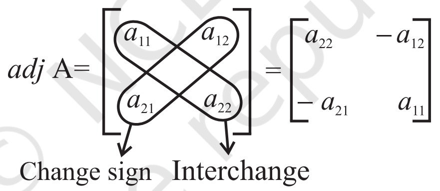
Solution We have $\text{A}_{11} = 4, \text{A}_{12} = -1, \text{A}_{21} = -3, \text{A}_{22} = 2$

Hence $$adj\, \text{A} = \begin{bmatrix} \text{A}_{11} & \text{A}_{21} \\ \text{A}_{12} & \text{A}_{22} \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$

Remark For a square matrix of order 2, given by

$$\text{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The *adj* A can also be obtained by interchanging a_{11} and a_{22} and by changing signs of a_{12} and a_{21}, i.e.,



We state the following theorem without proof.

Theorem 1 If A be any given square matrix of order n, then

$$\text{A}(adj\ \text{A}) = (adj\ \text{A})\,\text{A} = |\text{A}|\,\text{I},$$

where I is the identity matrix of order n

Verification

Let $$\text{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{ then } adj\ \text{A} = \begin{bmatrix} \text{A}_{11} & \text{A}_{21} & \text{A}_{31} \\ \text{A}_{12} & \text{A}_{22} & \text{A}_{32} \\ \text{A}_{13} & \text{A}_{23} & \text{A}_{33} \end{bmatrix}$$

Since sum of product of elements of a row (or a column) with corresponding cofactors is equal to |A| and otherwise zero, we have

$$A\,(adj\,A) = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\ I$$

Similarly, we can show $(adj\,A)\,A = |A|\ I$

Hence $A\,(adj\,A) = (adj\,A)\,A = |A|\ I$

Definition 4 A square matrix A is said to be singular if $|A| = 0$.

For example, the determinant of matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$ is zero

Hence A is a singular matrix.

Definition 5 A square matrix A is said to be non-singular if $|A| \neq 0$

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Then $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0$.

Hence A is a nonsingular matrix

We state the following theorems without proof.

Theorem 2 If A and B are nonsingular matrices of the same order, then AB and BA are also nonsingular matrices of the same order.

Theorem 3 The determinant of the product of matrices is equal to product of their respective determinants, that is, $|AB| = |A|\ |B|$, where A and B are square matrices of the same order

Remark We know that $(adj\,A)\,A = |A|\ I = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}, |A| \neq 0$

Writing determinants of matrices on both sides, we have

$$|(adj\,A)\,A| = \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix}$$

i.e. $$|(adj\ A)|\ |A| = |A|^3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad \text{(Why?)}$$

i.e. $$|(adj\ A)|\ |A| = |A|^3\ (1)$$

i.e. $$|(adj\ A)| = |A|^2$$

In general, if A is *a* square matrix of order *n*, then $|adj(A)| = |A|^{n-1}$.

Theorem 4 A square matrix A is invertible if and only if A is nonsingular matrix.

Proof Let A be invertible matrix of order *n* and I be the identity matrix of order *n*. Then, there exists a square matrix B of order *n* such that AB = BA = I

Now $\quad AB = I.\ \text{So } |AB| = |I| \ \text{ or } \ |A|\ |B| = 1 \quad (\text{since } |I|=1, |AB|=|A||B|)$

This gives $\quad |A| \neq 0$. Hence A is nonsingular.

Conversely, let A be nonsingular. Then $|A| \neq 0$

Now $\quad A\,(adj\ A) = (adj\ A)\,A = |A|\,I \qquad$ (Theorem 1)

or $$A\left(\frac{1}{|A|} adj\ A\right) = \left(\frac{1}{|A|} adj\ A\right) A = I$$

or $$AB = BA = I, \text{ where } B = \frac{1}{|A|} adj\ A$$

Thus $\quad$ A is invertible and $A^{-1} = \dfrac{1}{|A|} adj\ A$

Example 13 If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, then verify that A *adj* A = | A | I. Also find A^{-1}.

Solution We have $|A| = 1\,(16 - 9) - 3\,(4 - 3) + 3\,(3 - 4) = 1 \neq 0$

Now $A_{11} = 7$, $A_{12} = -1$, $A_{13} = -1$, $A_{21} = -3$, $A_{22} = 1$, $A_{23} = 0$, $A_{31} = -3$, $A_{32} = 0$, $A_{33} = 1$

Therefore $$adj\ A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Now $$A\,(adj\ A) = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7-3-3 & -3+3+0 & -3+0+3 \\ 7-4-3 & -3+4+0 & -3+0+3 \\ 7-3-4 & -3+3+0 & -3+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\,.\,I$$

Also $$A^{-1} = \frac{1}{|A|}\, adj\ A = \frac{1}{1}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Example 14 If $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, then verify that $(AB)^{-1} = B^{-1}A^{-1}$.

Solution We have $AB = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 5 & -14 \end{bmatrix}$

Since, $|AB| = -11 \neq 0$, $(AB)^{-1}$ exists and is given by

$$(AB)^{-1} = \frac{1}{|AB|}\, adj\,(AB) = -\frac{1}{11}\begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$$

Further, $|A| = -11 \neq 0$ and $|B| = 1 \neq 0$. Therefore, A^{-1} and B^{-1} both exist and are given by

$$A^{-1} = -\frac{1}{11}\begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix}, B^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

Therefore $$B^{-1}A^{-1} = -\frac{1}{11}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -4 & -3 \\ -1 & 2 \end{bmatrix} = -\frac{1}{11}\begin{bmatrix} -14 & -5 \\ -5 & -1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 14 & 5 \\ 5 & 1 \end{bmatrix}$$

Hence $(AB)^{-1} = B^{-1}\,A^{-1}$

Example 15 Show that the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies the equation $A^2 - 4A + I = O$, where I is 2 × 2 identity matrix and O is 2 × 2 zero matrix. Using this equation, find A^{-1}.

Solution We have $A^2 = A.A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$

Hence $A^2 - 4A + I = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

Now $A^2 - 4A + I = O$

Therefore $A A - 4A = -I$

or $A\ A\ (A^{-1}) - 4\ A\ A^{-1} = -I\ A^{-1}$ (Post multiplying by A^{-1} because $|A| \neq 0$)

or $A\ (A\ A^{-1}) - 4I = -A^{-1}$

or $AI - 4I = -A^{-1}$

or $A^{-1} = 4I - A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

Hence $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

EXERCISE 4.4

Find adjoint of each of the matrices in Exercises 1 and 2.

1. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **2.** $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$

Verify A (*adj* A) = (*adj* A) A = |A| I in Exercises 3 and 4

3. $\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$ **4.** $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

Find the inverse of each of the matrices (if it exists) given in Exercises 5 to 11.

5. $\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$ **6.** $\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$ **7.** $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$ **9.** $\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$ **10.** $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$

12. Let $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$. Verify that $(AB)^{-1} = B^{-1} A^{-1}$.

13. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$. Hence find A^{-1}.

14. For the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, find the numbers a and b such that $A^2 + aA + bI = O$.

15. For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

Show that $A^3 - 6A^2 + 5A + 11\,I = O$. Hence, find A^{-1}.

16. If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Verify that $A^3 - 6A^2 + 9A - 4I = O$ and hence find A^{-1}

17. Let A be a nonsingular square matrix of order 3 × 3. Then $|adj\ A|$ is equal to

(A) $|A|$ (B) $|A|^2$ (C) $|A|^3$ (D) $3|A|$

18. If A is an invertible matrix of order 2, then det (A^{-1}) is equal to

(A) det (A) (B) $\dfrac{1}{\det(A)}$ (C) 1 (D) 0

4.6 Applications of Determinants and Matrices

In this section, we shall discuss application of determinants and matrices for solving the system of linear equations in two or three variables and for checking the consistency of the system of linear equations.

Consistent system A system of equations is said to be *consistent* if its solution (one or more) exists.

Inconsistent system A system of equations is said to be *inconsistent* if its solution does not exist.

Note In this chapter, we restrict ourselves to the system of linear equations having unique solutions only.

4.6.1 *Solution of system of linear equations using inverse of a matrix*

Let us express the system of linear equations as matrix equations and solve them using inverse of the coefficient matrix.

Consider the system of equations

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

Let $$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Then, the system of equations can be written as, AX = B, i.e.,

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Case I If A is a nonsingular matrix, then its inverse exists. Now

$$AX = B$$

or $$A^{-1} (AX) = A^{-1} B \quad \text{(premultiplying by } A^{-1})$$

or $$(A^{-1}A) X = A^{-1} B \quad \text{(by associative property)}$$

or $$I X = A^{-1} B$$

or $$X = A^{-1} B$$

This matrix equation provides unique solution for the given system of equations as inverse of a matrix is unique. This method of solving system of equations is known as Matrix Method.

Case II If A is a singular matrix, then $|A| = 0$.

In this case, we calculate (*adj* A) B.

If (*adj* A) B ≠ O, (O being zero matrix), then solution does not exist and the system of equations is called inconsistent.

If (*adj* A) B = O, then system may be either consistent or inconsistent according as the system have either infinitely many solutions or no solution.

Example 16 Solve the system of equations

$$2x + 5y = 1$$
$$3x + 2y = 7$$

Solution The system of equations can be written in the form AX = B, where

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

Now, $|A| = -11 \neq 0$, Hence, A is nonsingular matrix and so has a unique solution.

Note that $$A^{-1} = -\frac{1}{11}\begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$$

Therefore $$X = A^{-1}B = -\frac{1}{11}\begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

i.e. $$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{11}\begin{bmatrix} -33 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Hence $$x = 3, y = -1$$

Example 17 Solve the following system of equations by matrix method.

$$3x - 2y + 3z = 8$$
$$2x + y - z = 1$$
$$4x - 3y + 2z = 4$$

Solution The system of equations can be written in the form AX = B, where

$$A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

We see that

$$|A| = 3\,(2 - 3) + 2(4 + 4) + 3\,(-6 - 4) = -17 \neq 0$$

Hence, A is nonsingular and so its inverse exists. Now

$$A_{11} = -1, \quad A_{12} = -8, \quad A_{13} = -10$$
$$A_{21} = -5, \quad A_{22} = -6, \quad A_{23} = 1$$
$$A_{31} = -1, \quad A_{32} = 9, \quad A_{33} = 7$$

Therefore $$A^{-1} = -\frac{1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

So $$X = A^{-1}B = -\frac{1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}\begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

i.e. $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{17}\begin{bmatrix} -17 \\ -34 \\ -51 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Hence $x = 1, y = 2$ and $z = 3$.

Example 18 The sum of three numbers is 6. If we multiply third number by 3 and add second number to it, we get 11. By adding first and third numbers, we get double of the second number. Represent it algebraically and find the numbers using matrix method.

Solution Let first, second and third numbers be denoted by x, y and z, respectively. Then, according to given conditions, we have

$$x + y + z = 6$$
$$y + 3z = 11$$
$$x + z = 2y \text{ or } x - 2y + z = 0$$

This system can be written as A X = B, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & 2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

Here $|A| = 1(1+6) - (0-3) + (0-1) = 9 \neq 0$. Now we find *adj* A

$A_{11} = 1(1 + 6) = 7,\quad A_{12} = -(0 - 3) = 3,\quad A_{13} = -1$
$A_{21} = -(1 + 2) = -3,\quad A_{22} = 0,\quad A_{23} = -(-2 - 1) = 3$
$A_{31} = (3 - 1) = 2,\quad A_{32} = -(3 - 0) = -3,\quad A_{33} = (1 - 0) = 1$

Hence $$adj\ A = \begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$$

Thus $$A^{-1} = \frac{1}{|A|} adj\,(A) = \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$$

Since $$X = A^{-1} B$$

$$X = \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

or $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 42-33+0 \\ 18+\ 0+0 \\ -6+33+0 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 \\ 18 \\ 27 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Thus $x = 1, y = 2, z = 3$

EXERCISE 4.5

Examine the consistency of the system of equations in Exercises 1 to 6.

1. $x + 2y = 2$
$2x + 3y = 3$

2. $2x - y = 5$
$x + y = 4$

3. $x + 3y = 5$
$2x + 6y = 8$

4. $x + y + z = 1$
$2x + 3y + 2z = 2$
$ax + ay + 2az = 4$

5. $3x - y - 2z = 2$
$2y - z = -1$
$3x - 5y = 3$

6. $5x - y + 4z = 5$
$2x + 3y + 5z = 2$
$5x - 2y + 6z = -1$

Solve system of linear equations, using matrix method, in Exercises 7 to 14.

7. $5x + 2y = 4$
$7x + 3y = 5$

8. $2x - y = -2$
$3x + 4y = 3$

9. $4x - 3y = 3$
$3x - 5y = 7$

10. $5x + 2y = 3$
$3x + 2y = 5$

11. $2x + y + z = 1$
$x - 2y - z = \frac{3}{2}$
$3y - 5z = 9$

12. $x - y + z = 4$
$2x + y - 3z = 0$
$x + y + z = 2$

13. $2x + 3y + 3z = 5$
$x - 2y + z = -4$
$3x - y - 2z = 3$

14. $x - y + 2z = 7$
$3x + 4y - 5z = -5$
$2x - y + 3z = 12$

15. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1}. Using A^{-1} solve the system of equations

$$2x - 3y + 5z = 11$$
$$3x + 2y - 4z = -5$$
$$x + y - 2z = -3$$

16. The cost of 4 kg onion, 3 kg wheat and 2 kg rice is ₹ 60. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is ₹ 90. The cost of 6 kg onion 2 kg wheat and 3 kg rice is ₹ 70. Find cost of each item per kg by matrix method.

Miscellaneous Examples

Example 19 Use product $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 2 & 4 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 9 & 2 & 3 \\ 6 & 1 & 2 \end{bmatrix}$ to solve the system of equations

$$x - y + 2z = 1$$
$$2y - 3z = 1$$
$$3x - 2y + 4z = 2$$

Solution Consider the product $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$

$$= \begin{bmatrix} -2-9+12 & 0-2+2 & 1+3-4 \\ 0+18-18 & 0+4-3 & 0-6+6 \\ -6-18+24 & 0-4+4 & 3+6-8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$

Now, given system of equations can be written, in matrix form, as follows

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

or
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 9 & 2 & 3 \\ 6 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2+0+2 \\ 9+2-6 \\ 6+1-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$$

Hence $x = 0$, $y = 5$ and $z = 3$

Miscellaneous Exercises on Chapter 4

1. Prove that the determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is independent of θ.

2. Evaluate $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$.

3. If $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, find $(AB)^{-1}$

4. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$. Verify that

(i) $[adj\ A]^{-1} = adj\ (A^{-1})$ (ii) $(A^{-1})^{-1} = A$

5. Evaluate $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

6. Evaluate $\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$

Using properties of determinants in Exercises 11 to 15, prove that:

7. Solve the system of equations

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4$$

$$\frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1$$

$$\frac{6}{x}+\frac{9}{y}-\frac{20}{z}=2$$

Choose the correct answer in Exercise 17 to 19.

8. If x, y, z are nonzero real numbers, then the inverse of matrix $A=\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ is

(A) $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$ (B) $xyz\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

(C) $\frac{1}{xyz}\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ (D) $\frac{1}{xyz}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

9. Let $A=\begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$, where $0 \le \theta \le 2\pi$. Then

(A) Det (A) = 0 (B) Det (A) $\in (2, \infty)$

(C) Det (A) $\in (2, 4)$ (D) Det (A) $\in [2, 4]$

Summary

- Determinant of a matrix $A = [a_{11}]_{1 \times 1}$ is given by $|a_{11}| = a_{11}$
- Determinant of a matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is given by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\, a_{22} - a_{12}\, a_{21}$$

- Determinant of a matrix $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is given by (expanding along R_1)

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

For any square matrix A, the |A| satisfy following properties.

- Area of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

- Minor of an element a_{ij} of the determinant of matrix A is the determinant obtained by deleting i^{th} row and j^{th} column and denoted by M_{ij}.
- Cofactor of a_{ij} of given by $A_{ij} = (-1)^{i+j} M_{ij}$
- Value of determinant of a matrix A is obtained by sum of product of elements of a row (or a column) with corresponding cofactors. For example,

$$|A| = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}.$$

- If elements of one row (or column) are multiplied with cofactors of elements of any other row (or column), then their sum is zero. For example, $a_{11} A_{21} + a_{12} A_{22} + a_{13} A_{23} = 0$

- If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then *adj* $A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$, where A_{ij} is cofactor of a_{ij}
- A (*adj* A) = (*adj* A) A = |A| I, where A is square matrix of order *n*.
- A square matrix A is said to be singular or non-singular according as |A| = 0 or |A| ≠ 0.
- If AB = BA = I, where B is square matrix, then B is called inverse of A. Also $A^{-1} = B$ or $B^{-1} = A$ and hence $(A^{-1})^{-1} = A$.
- A square matrix A has inverse if and only if A is non-singular.
- $A^{-1} = \frac{1}{|A|}(adj\ A)$
- If $a_1 x + b_1 y + c_1 z = d_1$
  $a_2 x + b_2 y + c_2 z = d_2$
  $a_3 x + b_3 y + c_3 z = d_3$,
  then these equations can be written as A X = B, where

  $$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

- Unique solution of equation AX = B is given by $X = A^{-1} B$, where $|A| \neq 0$.
- A system of equation is consistent or inconsistent according as its solution exists or not.
- For a square matrix A in matrix equation AX = B
  (i) |A| ≠ 0, there exists unique solution
  (ii) |A| = 0 and (*adj* A) B ≠ 0, then there exists no solution
  (iii) |A| = 0 and (*adj* A) B = 0, then system may or may not be consistent.

Historical Note

The Chinese method of representing the coefficients of the unknowns of several linear equations by using rods on a calculating board naturally led to the discovery of simple method of elimination. The arrangement of rods was precisely that of the numbers in a determinant. The Chinese, therefore, early developed the idea of subtracting columns and rows as in simplification of a determinant *Mikami, China*, *pp* 30, 93.

Seki Kowa, the greatest of the Japanese Mathematicians of seventeenth century in his work '*Kai Fukudai no Ho*' in 1683 showed that he had the idea of determinants and of their expansion. But he used this device only in eliminating a quantity from two equations and not directly in the solution of a set of simultaneous linear equations. T. Hayashi, "*The Fakudoi and Determinants in Japanese Mathematics*," in the proc. of the Tokyo Math. Soc., V.

Vendermonde was the first to recognise determinants as independent functions. He may be called the formal founder. Laplace (1772), gave general method of expanding a determinant in terms of its complementary minors. In 1773 Lagrange treated determinants of the second and third orders and used them for purpose other than the solution of equations. In 1801, Gauss used determinants in his theory of numbers.

The next great contributor was Jacques - Philippe - Marie Binet, (1812) who stated the theorem relating to the product of two matrices of m-columns and n-rows, which for the special case of $m = n$ reduces to the multiplication theorem.

Also on the same day, Cauchy (1812) presented one on the same subject. He used the word 'determinant' in its present sense. He gave the proof of multiplication theorem more satisfactory than Binet's.

The greatest contributor to the theory was Carl Gustav Jacob Jacobi, after this the word determinant received its final acceptance.

Chapter 5

CONTINUITY AND DIFFERENTIABILITY

❖ *The whole of science is nothing more than a refinement of everyday thinking." — ALBERT EINSTEIN* ❖

5.1 Introduction

This chapter is essentially a continuation of our study of differentiation of functions in Class XI. We had learnt to differentiate certain functions like polynomial functions and trigonometric functions. In this chapter, we introduce the very important concepts of continuity, differentiability and relations between them. We will also learn differentiation of inverse trigonometric functions. Further, we introduce a new class of functions called exponential and logarithmic functions. These functions lead to powerful techniques of differentiation. We illustrate certain geometrically obvious conditions through differential calculus. In the process, we will learn some fundamental theorems in this area.

Sir Issac Newton (1642-1727)

5.2 Continuity

We start the section with two informal examples to get a feel of continuity. Consider the function

$$f(x) = \begin{cases} 1, \text{ if } x \le 0 \\ 2, \text{ if } x > 0 \end{cases}$$

This function is of course defined at every point of the real line. Graph of this function is given in the Fig 5.1. One can deduce from the graph that the value of the function at *nearby* points on x-axis remain *close* to each other except at $x = 0$. At the points near and to the left of 0, i.e., at points like – 0.1, – 0.01, – 0.001, the value of the function is 1. At the points near and to the right of 0, i.e., at points like 0.1, 0.01,

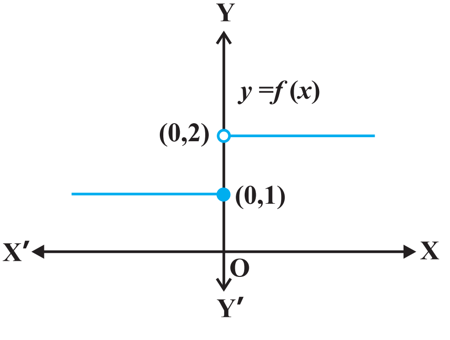


Fig 5.1

0.001, the value of the function is 2. Using the language of left and right hand limits, we may say that the left (respectively right) hand limit of f at 0 is 1 (respectively 2). In particular the left and right hand limits do not coincide. We also observe that the value of the function at $x = 0$ concides with the left hand limit. Note that when we try to draw the graph, we cannot draw it in one stroke, i.e., without lifting pen from the plane of the paper, we can not draw the graph of this function. In fact, we need to lift the pen when we come to 0 from left. This is one instance of function being not continuous at $x = 0$.

Now, consider the function defined as

$$f(x) = \begin{cases} 1, \text{ if } x \neq 0 \\ 2, \text{ if } x = 0 \end{cases}$$

This function is also defined at every point. Left and the right hand limits at $x = 0$ are both equal to 1. But the value of the function at $x = 0$ equals 2 which does not coincide with the common value of the left and right hand limits. Again, we note that we cannot draw the graph of the function without lifting the pen. This is yet another instance of a function being not continuous at $x = 0$.

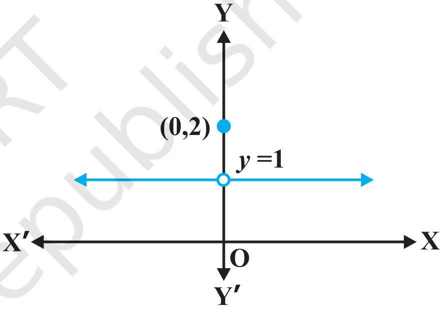


Fig 5.2

Naively, we may say that a function is continuous at a fixed point if we can draw the graph of the function *around* that point without lifting the pen from the plane of the paper.

Mathematically, it may be phrased precisely as follows:

Definition 1 Suppose f is a real function on a subset of the real numbers and let c be a point in the domain of f. Then f is continuous at c if

$$\lim_{x \to c} f(x) = f(c)$$

More elaborately, if the left hand limit, right hand limit and the value of the function at $x = c$ exist and equal to each other, then f is said to be continuous at $x = c$. Recall that if the right hand and left hand limits at $x = c$ coincide, then we say that the common value is the limit of the function at $x = c$. Hence we may also rephrase the definition of continuity as follows: *a function is continuous at $x = c$ if the function is defined at $x = c$ and if the value of the function at $x = c$ equals the limit of the function at $x = c$.* If f is not continuous at c, we say f is *discontinuous* at c and c is called a *point of discontinuity* of f.

Example 1 Check the continuity of the function f given by $f(x) = 2x + 3$ at $x = 1$.

Solution First note that the function is defined at the given point $x = 1$ and its value is 5. Then find the limit of the function at $x = 1$. Clearly

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}(2x+3) = 2(1)+3 = 5$$

Thus $$\lim_{x\to 1} f(x) = 5 = f(1)$$

Hence, f is continuous at $x = 1$.

Example 2 Examine whether the function f given by $f(x) = x^2$ is continuous at $x = 0$.

Solution First note that the function is defined at the given point $x = 0$ and its value is 0. Then find the limit of the function at $x = 0$. Clearly

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 = 0^2 = 0$$

Thus $$\lim_{x\to 0} f(x) = 0 = f(0)$$

Hence, f is continuous at $x = 0$.

Example 3 Discuss the continuity of the function f given by $f(x) = |\,x\,|$ at $x = 0$.

Solution By definition

$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Clearly the function is defined at 0 and $f(0) = 0$. Left hand limit of f at 0 is

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(-x) = 0$$

Similarly, the right hand limit of f at 0 is

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x = 0$$

Thus, the left hand limit, right hand limit and the value of the function coincide at $x = 0$. Hence, f is continuous at $x = 0$.

Example 4 Show that the function f given by

$$f(x) = \begin{cases} x^3 + 3, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$$

is not continuous at $x = 0$.

Solution The function is defined at $x = 0$ and its value at $x = 0$ is 1. When $x \neq 0$, the function is given by a polynomial. Hence,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (x^3 + 3) = 0^3 + 3 = 3$$

Since the limit of f at $x = 0$ does not coincide with $f(0)$, the function is not continuous at $x = 0$. It may be noted that $x = 0$ is the only point of discontinuity for this function.

Example 5 Check the points where the constant function $f(x) = k$ is continuous.

Solution The function is defined at all real numbers and by definition, its value at any real number equals k. Let c be any real number. Then

$$\lim_{x \to c} f(x) = \lim_{x \to c} k = k$$

Since $f(c) = k = \lim_{x \to c} f(x)$ for any real number c, the function f is continuous at every real number.

Example 6 Prove that the identity function on real numbers given by $f(x) = x$ is continuous at every real number.

Solution The function is clearly defined at every point and $f(c) = c$ for every real number c. Also,

$$\lim_{x \to c} f(x) = \lim_{x \to c} x = c$$

Thus, $\lim_{x \to c} f(x) = c = f(c)$ and hence the function is continuous at every real number.

Having defined continuity of a function at a given point, now we make a natural extension of this definition to discuss continuity of a function.

Definition 2 A real function f is said to be continuous if it is continuous at every point in the domain of f.

This definition requires a bit of elaboration. Suppose f is a function defined on a closed interval $[a, b]$, then for f to be continuous, it needs to be continuous at every point in $[a, b]$ including the end points a and b. Continuity of f at a means

$$\lim_{x \to a^+} f(x) = f(a)$$

and continuity of f at b means

$$\lim_{x \to b^-} f(x) = f(b)$$

Observe that $\lim_{x \to a^-} f(x)$ *and* $\lim_{x \to b^+} f(x)$ *do not make sense. As a consequence of this definition, if f is defined only at one point, it is continuous there, i.e., if the domain of f is a singleton, f is a continuous function.*

Example 7 Is the function defined by $f(x) = |x|$, a continuous function?

Solution We may rewrite f as

$$f(x) = \begin{cases} -x, \text{ if } x < 0 \\ x, \quad \text{if } x \geq 0 \end{cases}$$

By Example 3, we know that f is continuous at $x = 0$.

Let c be a real number such that $c < 0$. Then $f(c) = -c$. Also

$$\lim_{x \to c} f(x) = \lim_{x \to c} (-x) = -c \qquad \text{(Why?)}$$

Since $\lim_{x \to c} f(x) = f(c)$, f is continuous at all negative real numbers.

Now, let c be a real number such that $c > 0$. Then $f(c) = c$. Also

$$\lim_{x \to c} f(x) = \lim_{x \to c} x = c \qquad \text{(Why?)}$$

Since $\lim_{x \to c} f(x) = f(c)$, f is continuous at all positive real numbers. Hence, f is continuous at all points.

Example 8 Discuss the continuity of the function f given by $f(x) = x^3 + x^2 - 1$.

Solution Clearly f is defined at every real number c and its value at c is $c^3 + c^2 - 1$. We also know that

$$\lim_{x \to c} f(x) = \lim_{x \to c} (x^3 + x^2 - 1) = c^3 + c^2 - 1$$

Thus $\lim_{x \to c} f(x) = f(c)$, and hence f is continuous at every real number. This means f is a continuous function.

Example 9 Discuss the continuity of the function f defined by $f(x) = \dfrac{1}{x}$, $x \neq 0$.

Solution Fix any non zero real number c, we have

$$\lim_{x \to c} f(x) = \lim_{x \to c} \frac{1}{x} = \frac{1}{c}$$

Also, since for $c \neq 0$, $f(c) = \dfrac{1}{c}$, we have $\lim_{x \to c} f(x) = f(c)$ and hence, f is continuous at every point in the domain of f. Thus f is a continuous function.

We take this opportunity to explain the concept of *infinity*. This we do by analysing the function $f(x) = \frac{1}{x}$ near $x = 0$. To carry out this analysis we follow the usual trick of finding the value of the function at real numbers *close* to 0. Essentially we are trying to find the right hand limit of f at 0. We tabulate this in the following (Table 5.1).

Table 5.1

x	1	0.3	0.2	$0.1 = 10^{-1}$	$0.01 = 10^{-2}$	$0.001 = 10^{-3}$	10^{-n}
$f(x)$	1	3.333...	5	10	$100 = 10^2$	$1000 = 10^3$	10^n

We observe that as x gets closer to 0 from the right, the value of $f(x)$ shoots up higher. This may be rephrased as: the value of $f(x)$ may be made larger than any given number by choosing a positive real number *very close* to 0. In symbols, we write

$$\lim_{x \to 0^+} f(x) = +\infty$$

(to be read as: the right hand limit of $f(x)$ at 0 is plus infinity). We wish to emphasise that $+\infty$ is NOT a real number and hence the right hand limit of f at 0 does not exist (as a real number).

Similarly, the left hand limit of f at 0 may be found. The following table is self explanatory.

Table 5.2

x	-1	-0.3	-0.2	-10^{-1}	-10^{-2}	-10^{-3}	-10^{-n}
$f(x)$	-1	$-3.333...$	-5	-10	-10^2	-10^3	-10^n

From the Table 5.2, we deduce that the value of $f(x)$ may be made smaller than any given number by choosing a negative real number *very close* to 0. In symbols, we write

$$\lim_{x \to 0^-} f(x) = -\infty$$

(to be read as: the left hand limit of $f(x)$ at 0 is minus infinity). Again, we wish to emphasise that $-\infty$ is NOT a real number and hence the left hand limit of f at 0 does not exist (as a real number). The graph of the reciprocal function given in Fig 5.3 is a geometric representation of the above mentioned facts.

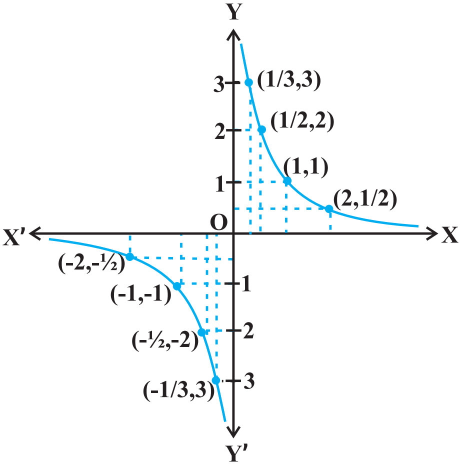


Fig 5.3

Example 10 Discuss the continuity of the function f defined by

$$f(x) = \begin{cases} x+2, \text{ if } x \le 1 \\ x-2, \text{ if } x > 1 \end{cases}$$

Solution The function f is defined at all points of the real line.

Case 1 If $c < 1$, then $f(c) = c + 2$. Therefore, $\lim_{x \to c} f(x) = \lim_{x \to c} (x+2) = c+2$

Thus, f is continuous at all real numbers less than 1.

Case 2 If $c > 1$, then $f(c) = c - 2$. Therefore,

$$\lim_{x \to c} f(x) = \lim_{x \to c} (x-2) = c-2 = f(c)$$

Thus, f is continuous at all points $x > 1$.

Case 3 If $c = 1$, then the left hand limit of f at $x = 1$ is

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x+2) = 1+2 = 3$$

The right hand limit of f at $x = 1$ is

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x-2) = 1-2 = -1$$

Since the left and right hand limits of f at $x = 1$ do not coincide, f is not continuous at $x = 1$. Hence $x = 1$ is the only point of discontinuity of f. The graph of the function is given in Fig 5.4.

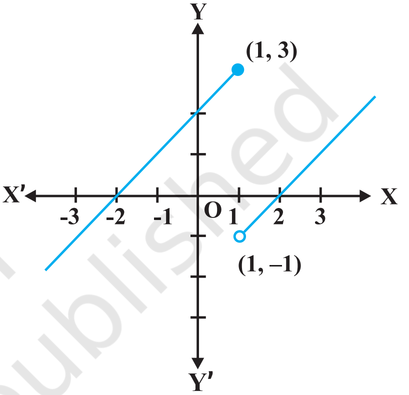


Fig 5.4

Example 11 Find all the points of discontinuity of the function f defined by

$$f(x) = \begin{cases} x+2, \text{ if } x < 1 \\ 0, \quad \text{ if } x = 1 \\ x-2, \text{ if } x > 1 \end{cases}$$

Solution As in the previous example we find that f is continuous at all real numbers $x \neq 1$. The left hand limit of f at $x = 1$ is

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x+2) = 1+2 = 3$$

The right hand limit of f at $x = 1$ is

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x-2) = 1-2 = -1$$

Since, the left and right hand limits of f at $x = 1$ do not coincide, f is not continuous at $x = 1$. Hence $x = 1$ is the only point of discontinuity of f. The graph of the function is given in the Fig 5.5.

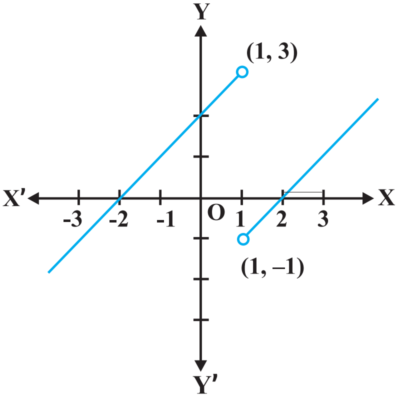


Fig 5.5

Example 12 Discuss the continuity of the function defined by

$$f(x) = \begin{cases} x+2, \text{ if } x<0 \\ -x+2, \text{ if } x>0 \end{cases}$$

Solution Observe that the function is defined at all real numbers except at 0. Domain of definition of this function is

$D_1 \cup D_2$ where $D_1 = \{x \in \mathbf{R} : x < 0\}$ and
$D_2 = \{x \in \mathbf{R} : x > 0\}$

Case 1 If $c \in D_1$, then $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} (x + 2)$ $= c + 2 = f(c)$ and hence f is continuous in D_1.

Case 2 If $c \in D_2$, then $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} (-x + 2)$ $= -c + 2 = f(c)$ and hence f is continuous in D_2.

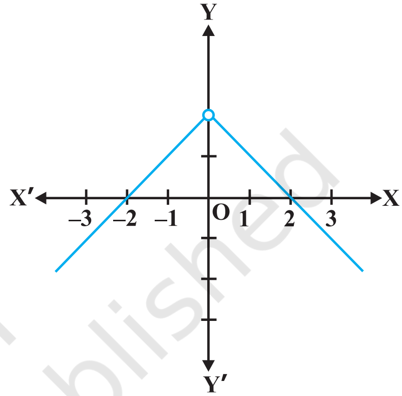


Fig 5.6

Since f is continuous at all points in the domain of f, we deduce that f is continuous. Graph of this function is given in the Fig 5.6. Note that to graph this function we need to lift the pen from the plane of the paper, but we need to do that only for those points where the function is not defined.

Example 13 Discuss the continuity of the function f given by

$$f(x) = \begin{cases} x, & \text{if } x \geq 0 \\ x^2, & \text{if } x < 0 \end{cases}$$

Solution Clearly the function is defined at every real number. Graph of the function is given in Fig 5.7. By inspection, it seems prudent to partition the domain of definition of f into three disjoint subsets of the real line.

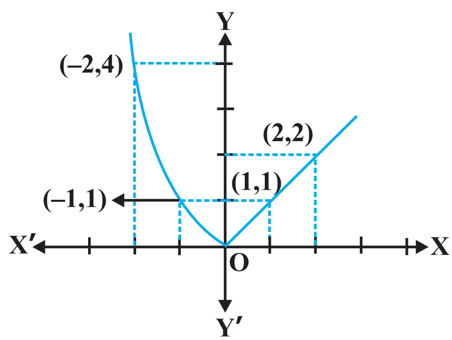


Fig 5.7

Let $D_1 = \{x \in \mathbf{R} : x < 0\}$, $D_2 = \{0\}$ and
$D_3 = \{x \in \mathbf{R} : x > 0\}$

Case 1 At any point in D_1, we have $f(x) = x^2$ and it is easy to see that it is continuous there (see Example 2).

Case 2 At any point in D_3, we have $f(x) = x$ and it is easy to see that it is continuous there (see Example 6).

Case 3 Now we analyse the function at $x = 0$. The value of the function at 0 is $f(0) = 0$. The left hand limit of f at 0 is

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 = 0^2 = 0$$

The right hand limit of f at 0 is

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0$$

Thus $\lim_{x \to 0} f(x) = 0 = f(0)$ and hence f is continuous at 0. This means that f is continuous at every point in its domain and hence, f is a continuous function.

Example 14 Show that every polynomial function is continuous.

Solution Recall that a function p is a polynomial function if it is defined by $p(x) = a_0 + a_1 x + ... + a_n x^n$ for some natural number n, $a_n \neq 0$ and $a_i \in \mathbf{R}$. Clearly this function is defined for every real number. For a fixed real number c, we have

$$\lim_{x \to c} p(x) = p(c)$$

By definition, p is continuous at c. Since c is any real number, p is continuous at every real number and hence p is a continuous function.

Example 15 Find all the points of discontinuity of the greatest integer function defined by $f(x) = [x]$, where $[x]$ denotes the greatest integer less than or equal to x.

Solution First observe that f is defined for all real numbers. Graph of the function is given in Fig 5.8. From the graph it looks like that f is discontinuous at every integral point. Below we explore, if this is true.

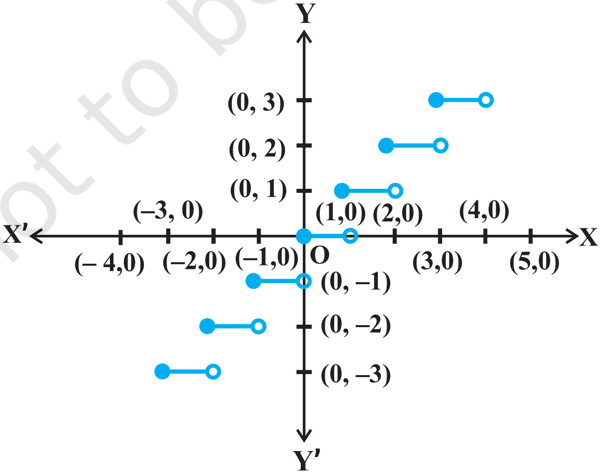


Fig 5.8

Case 1 Let c be a real number which is not equal to any integer. It is evident from the graph that for all real numbers *close* to c the value of the function is equal to $[c]$; i.e., $\lim_{x \to c} f(x) = \lim_{x \to c} [x] = [c]$. Also $f(c) = [c]$ and hence the function is continuous at all real numbers not equal to integers.

Case 2 Let c be an integer. Then we can find a sufficiently small real number $r > 0$ such that $[c - r] = c - 1$ whereas $[c + r] = c$.

This, in terms of limits mean that

$$\lim_{x \to c^-} f(x) = c - 1, \quad \lim_{x \to c^+} f(x) = c$$

Since these limits cannot be equal to each other for any c, the function is discontinuous at every integral point.

5.2.1 Algebra of continuous functions

In the previous class, after having understood the concept of limits, we learnt some algebra of limits. Analogously, now we will study some algebra of continuous functions. Since continuity of a function at a point is entirely dictated by the limit of the function at that point, it is reasonable to expect results analogous to the case of limits.

Theorem 1 Suppose f and g be two real functions continuous at a real number c. Then

(1) $f + g$ is continuous at $x = c$.

(2) $f - g$ is continuous at $x = c$.

(3) $f \,.\, g$ is continuous at $x = c$.

(4) $\left(\frac{f}{g}\right)$ is continuous at $x = c$, (provided $g(c) \neq 0$).

Proof We are investigating continuity of $(f + g)$ at $x = c$. Clearly it is defined at $x = c$. We have

$$\begin{aligned} \lim_{x \to c}(f + g)(x) &= \lim_{x \to c} [f(x) + g(x)] && \text{(by definition of } f + g) \\ &= \lim_{x \to c} f(x) + \lim_{x \to c} g(x) && \text{(by the theorem on limits)} \\ &= f(c) + g(c) && \text{(as } f \text{ and } g \text{ are continuous)} \\ &= (f + g)(c) && \text{(by definition of } f + g) \end{aligned}$$

Hence, $f + g$ is continuous at $x = c$.

Proofs for the remaining parts are similar and left as an exercise to the reader.

Remarks

(i) As a special case of (3) above, if f is a constant function, i.e., $f(x) = \lambda$ for some real number λ, then the function $(\lambda \,.\, g)$ defined by $(\lambda \,.\, g)(x) = \lambda \,.\, g(x)$ is also continuous. In particular if $\lambda = -1$, the continuity of f implies continuity of $-f$.

(ii) As a special case of (4) above, if f is the constant function $f(x) = \lambda$, then the function $\frac{\lambda}{g}$ defined by $\frac{\lambda}{g}(x) = \frac{\lambda}{g(x)}$ is also continuous wherever $g(x) \neq 0$. In particular, the continuity of g implies continuity of $\frac{1}{g}$.

The above theorem can be exploited to generate many continuous functions. They also aid in deciding if certain functions are continuous or not. The following examples illustrate this:

Example 16 Prove that every rational function is continuous.

Solution Recall that every rational function f is given by

$$f(x) = \frac{p(x)}{q(x)}, \quad q(x) \neq 0$$

where p and q are polynomial functions. The domain of f is all real numbers except points at which q is zero. Since polynomial functions are continuous (Example 14), f is continuous by (4) of Theorem 1.

Example 17 Discuss the continuity of sine function.

Solution To see this we use the following facts

$$\lim_{x \to 0} \sin x = 0$$

We have not proved it, but is intuitively clear from the graph of $\sin x$ near 0.

Now, observe that $f(x) = \sin x$ is defined for every real number. Let c be a real number. Put $x = c + h$. If $x \to c$ we know that $h \to 0$. Therefore

$$\begin{aligned}
\lim_{x \to c} f(x) &= \lim_{x \to c} \sin x \\
&= \lim_{h \to 0} \sin(c + h) \\
&= \lim_{h \to 0} [\sin c \cos h + \cos c \sin h] \\
&= \lim_{h \to 0} [\sin c \cos h] + \lim_{h \to 0} [\cos c \sin h] \\
&= \sin c + 0 = \sin c = f(c)
\end{aligned}$$

Thus $\lim_{x \to c} f(x) = f(c)$ and hence f is a continuous function.

Remark A similar proof may be given for the continuity of cosine function.

Example 18 Prove that the function defined by $f(x) = \tan x$ is a continuous function.

Solution The function $f(x) = \tan x = \dfrac{\sin x}{\cos x}$. This is defined for all real numbers such that $\cos x \neq 0$, i.e., $x \neq (2n+1)\dfrac{\pi}{2}$. We have just proved that both sine and cosine functions are continuous. Thus tan x being a quotient of two continuous functions is continuous wherever it is defined.

An interesting fact is the behaviour of continuous functions with respect to composition of functions. Recall that if f and g are two real functions, then

$$(f \text{ o } g)\ (x) = f(g(x))$$

is defined whenever the range of g is a subset of domain of f. The following theorem (stated without proof) captures the continuity of composite functions.

Theorem 2 Suppose f and g are real valued functions such that $(f \text{ o } g)$ is defined at c. If g is continuous at c and if f is continuous at $g(c)$, then $(f \text{ o } g)$ is continuous at c.

The following examples illustrate this theorem.

Example 19 Show that the function defined by $f(x) = \sin(x^2)$ is a continuous function.

Solution Observe that the function is defined for every real number. The function f may be thought of as a composition g o h of the two functions g and h, where $g(x) = \sin x$ and $h(x) = x^2$. Since both g and h are continuous functions, by Theorem 2, it can be deduced that f is a continuous function.

Example 20 Show that the function f defined by

$$f(x) = |1 - x + |x||,$$

where x is any real number, is a continuous function.

Solution Define g by $g(x) = 1 - x + |x|$ and h by $h(x) = |x|$ for all real x. Then

$$\begin{aligned}(h \text{ o } g)\ (x) &= h(g(x)) \\ &= h(1 - x + |x|) \\ &= |1 - x + |x|| = f(x)\end{aligned}$$

In Example 7, we have seen that h is a continuous function. Hence g being a sum of a polynomial function and the modulus function is continuous. But then f being a composite of two continuous functions is continuous.

EXERCISE 5.1

1. Prove that the function $f(x) = 5x - 3$ is continuous at $x = 0$, at $x = -3$ and at $x = 5$.

2. Examine the continuity of the function $f(x) = 2x^2 - 1$ at $x = 3$.

3. Examine the following functions for continuity.

(a) $f(x) = x - 5$ (b) $f(x) = \dfrac{1}{x-5}, x \neq 5$

(c) $f(x) = \dfrac{x^2 - 25}{x+5}, x \neq -5$ (d) $f(x) = |x - 5|$

4. Prove that the function $f(x) = x^n$ is continuous at $x = n$, where n is a positive integer.

5. Is the function f defined by

$$f(x) = \begin{cases} x, & \text{if } x \leq 1 \\ 5, & \text{if } x > 1 \end{cases}$$

continuous at $x = 0$? At $x = 1$? At $x = 2$?

Find all points of discontinuity of f, where f is defined by

6. $f(x) = \begin{cases} 2x+3, & \text{if } x \leq 2 \\ 2x-3, & \text{if } x > 2 \end{cases}$

7. $f(x) = \begin{cases} |x|+3, & \text{if } x \leq -3 \\ -2x, & \text{if } -3 < x < 3 \\ 6x+2, & \text{if } x \geq 3 \end{cases}$

8. $f(x) = \begin{cases} \dfrac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

9. $f(x) = \begin{cases} \dfrac{x}{|x|}, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$

10. $f(x) = \begin{cases} x+1, & \text{if } x \geq 1 \\ x^2+1, & \text{if } x < 1 \end{cases}$

11. $f(x) = \begin{cases} x^3-3, & \text{if } x \leq 2 \\ x^2+1, & \text{if } x > 2 \end{cases}$

12. $f(x) = \begin{cases} x^{10}-1, & \text{if } x \leq 1 \\ x^2, & \text{if } x > 1 \end{cases}$

13. Is the function defined by

$$f(x) = \begin{cases} x+5, & \text{if } x \leq 1 \\ x-5, & \text{if } x > 1 \end{cases}$$

a continuous function?

Discuss the continuity of the function f, where f is defined by

14. $f(x) = \begin{cases} 3, & \text{if } 0 \le x \le 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \le x \le 10 \end{cases}$

15. $f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \le x \le 1 \\ 4x, & \text{if } x > 1 \end{cases}$

16. $f(x) = \begin{cases} -2, & \text{if } x \le -1 \\ 2x, & \text{if } -1 < x \le 1 \\ 2, & \text{if } x > 1 \end{cases}$

17. Find the relationship between a and b so that the function f defined by

$$f(x) = \begin{cases} ax + 1, & \text{if } x \le 3 \\ bx + 3, & \text{if } x > 3 \end{cases}$$

is continuous at $x = 3$.

18. For what value of λ is the function defined by

$$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \le 0 \\ 4x + 1, & \text{if } x > 0 \end{cases}$$

continuous at $x = 0$? What about continuity at $x = 1$?

19. Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral points. Here $[x]$ denotes the greatest integer less than or equal to x.

20. Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at $x = \pi$?

21. Discuss the continuity of the following functions:

(a) $f(x) = \sin x + \cos x$ (b) $f(x) = \sin x - \cos x$

(c) $f(x) = \sin x \, . \cos x$

22. Discuss the continuity of the cosine, cosecant, secant and cotangent functions.

23. Find all points of discontinuity of f, where

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x < 0 \\ x + 1, & \text{if } x \ge 0 \end{cases}$$

24. Determine if f defined by

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & \text{if } x \ne 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is a continuous function?

25. Examine the continuity of f, where f is defined by

$$f(x)=\begin{cases}\sin x-\cos x, & \text{if } x\neq 0\\ -1, & \text{if } x=0\end{cases}$$

Find the values of k so that the function f is continuous at the indicated point in Exercises 26 to 29.

26. $f(x)=\begin{cases}\dfrac{k\cos x}{\pi-2x}, & \text{if } x\neq\dfrac{\pi}{2}\\ 3, & \text{if } x=\dfrac{\pi}{2}\end{cases}$ at $x=\dfrac{\pi}{2}$

27. $f(x)=\begin{cases}kx^2, & \text{if } x\leq 2\\ 3, & \text{if } x>2\end{cases}$ at $x=2$

28. $f(x)=\begin{cases}kx+1, & \text{if } x\leq\pi\\ \cos x, & \text{if } x>\pi\end{cases}$ at $x=\pi$

29. $f(x)=\begin{cases}kx+1, & \text{if } x\leq 5\\ 3x-5, & \text{if } x>5\end{cases}$ at $x=5$

30. Find the values of a and b such that the function defined by

$$f(x)=\begin{cases}5, & \text{if } x\leq 2\\ ax+b, & \text{if } 2<x<10\\ 21, & \text{if } x\geq 10\end{cases}$$

is a continuous function.

31. Show that the function defined by $f(x) = \cos(x^2)$ is a continuous function.

32. Show that the function defined by $f(x) = |\cos x|$ is a continuous function.

33. Examine that $\sin|x|$ is a continuous function.

34. Find all the points of discontinuity of f defined by $f(x) = |x| - |x+1|$.

5.3. Differentiability

Recall the following facts from previous class. We had defined the derivative of a real function as follows:

Suppose f is a real function and c is a point in its domain. The derivative of f at c is defined by

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

provided this limit exists. Derivative of f at c is denoted by $f'(c)$ or $\frac{d}{dx}(f(x))|_c$. The function defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

wherever the limit exists is defined to be the derivative of f. The derivative of f is denoted by $f'(x)$ or $\frac{d}{dx}(f(x))$ or if $y = f(x)$ by $\frac{dy}{dx}$ or y'. The process of finding derivative of a function is called differentiation. We also use the phrase *differentiate* $f(x)$ *with respect to* x to mean *find* $f'(x)$.

The following rules were established as a part of algebra of derivatives:

(1) $(u \pm v)' = u' \pm v'$

(2) $(uv)' = u'v + uv'$ (Leibnitz or product rule)

(3) $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$, wherever $v \neq 0$ (Quotient rule).

The following table gives a list of derivatives of certain standard functions:

Table 5.3

$f(x)$	x^n	$\sin x$	$\cos x$	$\tan x$
$f'(x)$	nx^{n-1}	$\cos x$	$-\sin x$	$\sec^2 x$

Whenever we defined derivative, we had put a caution *provided the limit exists*. Now the natural question is; what if it doesn't? The question is quite pertinent and so is its answer. If $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ does not exist, we say that f is not differentiable at c. In other words, we say that a function f is differentiable at a point c in its domain if both $\lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}$ and $\lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$ are finite and equal. A function is said to be differentiable in an interval $[a, b]$ if it is differentiable at every point of $[a, b]$. As in case of continuity, at the end points a and b, we take the right hand limit and left hand limit, which are nothing but left hand derivative and right hand derivative of the function at a and b respectively. Similarly, a function is said to be differentiable in an interval (a, b) if it is differentiable at every point of (a, b).

Theorem 3 If a function f is differentiable at a point c, then it is also continuous at that point.

Proof Since f is differentiable at c, we have

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = f'(c)$$

But for $x \neq c$, we have

$$f(x) - f(c) = \frac{f(x)-f(c)}{x-c} \cdot (x-c)$$

Therefore $$\lim_{x\to c}[f(x)-f(c)] = \lim_{x\to c}\left[\frac{f(x)-f(c)}{x-c}\cdot(x-c)\right]$$

or $$\lim_{x\to c}[f(x)] - \lim_{x\to c}[f(c)] = \lim_{x\to c}\left[\frac{f(x)-f(c)}{x-c}\right]\cdot \lim_{x\to c}[(x-c)]$$

$$= f'(c) \cdot 0 = 0$$

or $$\lim_{x\to c} f(x) = f(c)$$

Hence f is continuous at $x = c$.

Corollary 1 Every differentiable function is continuous.

We remark that the converse of the above statement is not true. Indeed we have seen that the function defined by $f(x) = |x|$ is a continuous function. Consider the left hand limit

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \frac{-h}{h} = -1$$

The right hand limit

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \frac{h}{h} = 1$$

Since the above left and right hand limits at 0 are not equal, $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$ does not exist and hence f is not differentiable at 0. Thus f is not a differentiable function.

5.3.1 *Derivatives of composite functions*

To study derivative of composite functions, we start with an illustrative example. Say, we want to find the derivative of f, where

$$f(x) = (2x + 1)^3$$

One way is to expand $(2x + 1)^3$ using binomial theorem and find the derivative as a polynomial function as illustrated below.

$$\frac{d}{dx}f(x) = \frac{d}{dx}\left[(2x+1)^3\right]$$

$$= \frac{d}{dx}(8x^3+12x^2+6x+1)$$

$$= 24x^2 + 24x + 6$$

$$= 6\,(2x + 1)^2$$

Now, observe that $f(x) = (h \text{ o } g)(x)$

where $g(x) = 2x + 1$ and $h(x) = x^3$. Put $t = g(x) = 2x + 1$. Then $f(x) = h(t) = t^3$. Thus

$$\frac{df}{dx} = 6\,(2x + 1)^2 = 3(2x + 1)^2 \,.\, 2 = 3t^2 \,.\, 2 = \frac{dh}{dt}\cdot\frac{dt}{dx}$$

The advantage with such observation is that it simplifies the calculation in finding the derivative of, say, $(2x + 1)^{100}$. We may formalise this observation in the following theorem called the chain rule.

Theorem 4 **(Chain Rule)** Let f be a real valued function which is a composite of two functions u and v; i.e., $f = v \text{ o } u$. Suppose $t = u(x)$ and if both $\frac{dt}{dx}$ and $\frac{dv}{dt}$ exist, we have

$$\frac{df}{dx} = \frac{dv}{dt}\cdot\frac{dt}{dx}$$

We skip the proof of this theorem. Chain rule may be extended as follows. Suppose f is a real valued function which is a composite of three functions u, v and w; i.e.,

$f = (w \text{ o } u) \text{ o } v$. If $t = v(x)$ and $s = u(t)$, then

$$\frac{df}{dx} = \frac{d(w \text{o} u)}{dt}\cdot\frac{dt}{dx} = \frac{dw}{ds}\cdot\frac{ds}{dt}\cdot\frac{dt}{dx}$$

provided all the derivatives in the statement exist. Reader is invited to formulate chain rule for composite of more functions.

Example 21 Find the derivative of the function given by $f(x) = \sin(x^2)$.

Solution Observe that the given function is a composite of two functions. Indeed, if $t = u(x) = x^2$ and $v(t) = \sin t$, then

$$f(x) = (v \text{ o } u)(x) = v(u(x)) = v(x^2) = \sin x^2$$

Put $t = u(x) = x^2$. Observe that $\frac{dv}{dt} = \cos t$ and $\frac{dt}{dx} = 2x$ exist. Hence, by chain rule

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos t \cdot 2x$$

It is normal practice to express the final result only in terms of x. Thus

$$\frac{df}{dx} = \cos t \cdot 2x = 2x \cos x^2$$

EXERCISE 5.2

Differentiate the functions with respect to x in Exercises 1 to 8.

1. $\sin (x^2 + 5)$ **2.** $\cos (\sin x)$ **3.** $\sin (ax + b)$

4. $\sec (\tan (\sqrt{x}))$ **5.** $\frac{\sin (ax+b)}{\cos (cx+d)}$ **6.** $\cos x^3 . \sin^2 (x^5)$

7. $2\sqrt{\cot(x^2)}$ **8.** $\cos(\sqrt{x})$

9. Prove that the function f given by

$$f(x) = |x - 1|, x \in \mathbf{R}$$

is not differentiable at $x = 1$.

10. Prove that the greatest integer function defined by

$$f(x) = [x], 0 < x < 3$$

is not differentiable at $x = 1$ and $x = 2$.

5.3.2 *Derivatives of implicit functions*

Until now we have been differentiating various functions given in the form $y = f(x)$. But it is not necessary that functions are always expressed in this form. For example, consider one of the following relationships between x and y:

$$x - y - \pi = 0$$

$$x + \sin xy - y = 0$$

In the first case, we can *solve for y* and rewrite the relationship as $y = x - \pi$. In the second case, it does not seem that there is an easy way to *solve for y*. Nevertheless, there is no doubt about the dependence of y on x in either of the cases. When a relationship between x and y is expressed in a way that it is easy to *solve for y* and write $y = f(x)$, we say that y is given as an *explicit function* of x. In the latter case it is implicit that y is a function of x and we say that the relationship of the second type, above, gives function *implicitly*. In this subsection, we learn to differentiate implicit functions.

Example 22 Find $\frac{dy}{dx}$ if $x - y = \pi$.

Solution One way is to solve for y and rewrite the above as

$$y = x - \pi$$

But then $$\frac{dy}{dx} = 1$$

Alternatively, *directly* differentiating the relationship w.r.t., x, we have

$$\frac{d}{dx}(x-y) = \frac{d\pi}{dx}$$

Recall that $\frac{d\pi}{dx}$ means to differentiate the constant function taking value π everywhere w.r.t., x. Thus

$$\frac{d}{dx}(x) - \frac{d}{dx}(y) = 0$$

which implies that

$$\frac{dy}{dx} = \frac{dx}{dx} = 1$$

Example 23 Find $\frac{dy}{dx}$, if $y + \sin y = \cos x$.

Solution We differentiate the relationship directly with respect to x, i.e.,

$$\frac{dy}{dx} + \frac{d}{dx}(\sin y) = \frac{d}{dx}(\cos x)$$

which implies using chain rule

$$\frac{dy}{dx} + \cos y \cdot \frac{dy}{dx} = -\sin x$$

This gives $$\frac{dy}{dx} = -\frac{\sin x}{1+\cos y}$$

where $$y \neq (2n + 1)\pi$$

5.3.3 *Derivatives of inverse trigonometric functions*

We remark that inverse trigonometric functions are continuous functions, but we will not prove this. Now we use chain rule to find derivatives of these functions.

Example 24 Find the derivative of f given by $f(x) = \sin^{-1} x$ assuming it exists.

Solution Let $y = \sin^{-1} x$. Then, $x = \sin y$.

Differentiating both sides w.r.t. x, we get

$$1 = \cos y \frac{dy}{dx}$$

which implies that $$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos(\sin^{-1} x)}$$

Observe that this is defined only for $\cos y \neq 0$, i.e., $\sin^{-1} x \neq -\frac{\pi}{2}, \frac{\pi}{2}$, i.e., $x \neq -1, 1$, i.e., $x \in (-1, 1)$.

To make this result a bit more attractive, we carry out the following manipulation. Recall that for $x \in (-1, 1)$, $\sin(\sin^{-1} x) = x$ and hence

$$\cos^2 y = 1 - (\sin y)^2 = 1 - (\sin(\sin^{-1} x))^2 = 1 - x^2$$

Also, since $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\cos y$ is positive and hence $\cos y = \sqrt{1 - x^2}$

Thus, for $x \in (-1, 1)$,

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$$

$f(x)$	$\sin^{-1} x$	$\cos^{-1} x$	$\tan^{-1} x$
$f^1(x)$	$1/\sqrt{1-x^2}$	$-1/\sqrt{1-x^2}$	$1/1+x^2$
Domain off	(-1, 1)	(-1, 1)	R

EXERCISE 5.3

Find $\frac{dy}{dx}$ in the following:

1. $2x + 3y = \sin x$
2. $2x + 3y = \sin y$
3. $ax + by^2 = \cos y$
4. $xy + y^2 = \tan x + y$
5. $x^2 + xy + y^2 = 100$
6. $x^3 + x^2y + xy^2 + y^3 = 81$
7. $\sin^2 y + \cos xy = \kappa$
8. $\sin^2 x + \cos^2 y = 1$
9. $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$
10. $y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right), \; -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$
11. $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$
12. $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$
13. $y = \cos^{-1}\left(\frac{2x}{1+x^2}\right), -1 < x < 1$
14. $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right), -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$
15. $y = \sec^{-1}\left(\frac{1}{2x^2 - 1}\right), 0 < x < \frac{1}{\sqrt{2}}$

5.4 Exponential and Logarithmic Functions

Till now we have learnt some aspects of different classes of functions like polynomial functions, rational functions and trigonometric functions. In this section, we shall learn about a new class of (related) functions called exponential functions and logarithmic functions. It needs to be emphasized that many statements made in this section are motivational and precise proofs of these are well beyond the scope of this text.

The Fig 5.9 gives a sketch of $y = f_1(x) = x$, $y = f_2(x) = x^2$, $y = f_3(x) = x^3$ and $y = f_4(x) = x^4$. Observe that the curves get steeper as the power of x increases. Steeper the curve, faster is the rate of growth. What this means is that for a fixed increment in the value of x (> 1), the increment in the value of $y = f_n(x)$ increases as n increases for n = 1, 2, 3, 4. It is conceivable that such a statement is true for all positive values of n, where $f_n(x) = x^n$. Essentially, this means that the graph of $y = f_n(x)$ *leans* more towards the y-axis as n increases. For example, consider $f_{10}(x) = x^{10}$ and $f_{15}(x) = x^{15}$. If x increases from 1 to 2, f_{10} increases from 1 to 2^{10} whereas f_{15} increases from 1 to 2^{15}. Thus, for the same increment in x, f_{15} grow faster than f_{10}.

Upshot of the above discussion is that the growth of polynomial functions is dependent on the degree of the polynomial function – higher the degree, greater is the growth. The next natural question is: Is there a function which grows faster than any polynomial function. The answer is in affirmative and an example of such a function is

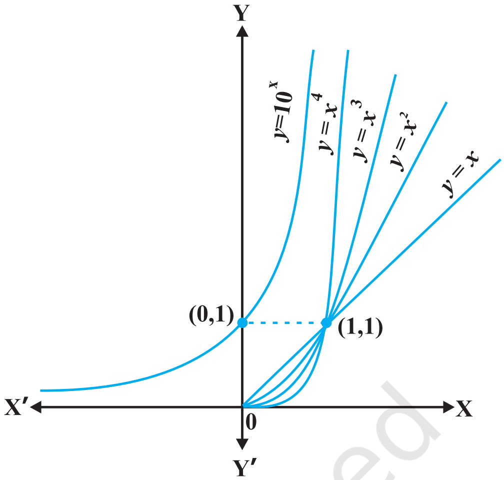


Fig 5.9

$$y = f(x) = 10^x.$$

Our claim is that this function f grows faster than $f_n(x) = x^n$ for any positive integer n. For example, we can prove that 10^x grows faster than $f_{100}(x) = x^{100}$. For large values of x like $x = 10^3$, note that $f_{100}(x) = (10^3)^{100} = 10^{300}$ whereas $f(10^3) = 10^{10^3} = 10^{1000}$. Clearly $f(x)$ is much greater than $f_{100}(x)$. It is not difficult to prove that for all $x > 10^3$, $f(x) > f_{100}(x)$. But we will not attempt to give a proof of this here. Similarly, by choosing large values of x, one can verify that $f(x)$ grows faster than $f_n(x)$ for any positive integer n.

Definition 3 The exponential function with positive base $b > 1$ is the function

$$y = f(x) = b^x$$

The graph of $y = 10^x$ is given in the Fig 5.9.

It is advised that the reader plots this graph for particular values of b like 2, 3 and 4. Following are some of the salient features of the exponential functions:

(1) Domain of the exponential function is **R**, the set of all real numbers.

(2) Range of the exponential function is the set of all positive real numbers.

(3) The point (0, 1) is always on the graph of the exponential function (this is a restatement of the fact that $b^0 = 1$ for any real $b > 1$).

(4) Exponential function is ever increasing; i.e., as we move from left to right, the graph rises above.

(5) For very large negative values of x, the exponential function is very close to 0. In other words, in the second quadrant, the graph approaches x-axis (but never meets it).

Exponential function with base 10 is called the *common exponential function*. In the Appendix A.1.4 of Class XI, it was observed that the sum of the series

$$1+\frac{1}{1!}+\frac{1}{2!}+...$$

is a number between 2 and 3 and is denoted by e. Using this e as the base we obtain an extremely important exponential function $y = e^x$.

This is called *natural exponential function*.

It would be interesting to know if the inverse of the exponential function exists and has *nice* interpretation. This search motivates the following definition.

Definition 4 Let $b > 1$ be a real number. Then we say logarithm of a to base b is x if $b^x = a$.

Logarithm of a to base b is denoted by $\log_b a$. Thus $\log_b a = x$ if $b^x = a$. Let us work with a few explicit examples to get a feel for this. We know $2^3 = 8$. In terms of logarithms, we may rewrite this as $\log_2 8 = 3$. Similarly, $10^4 = 10000$ is equivalent to saying $\log_{10} 10000 = 4$. Also, $625 = 5^4 = 25^2$ is equivalent to saying $\log_5 625 = 4$ or $\log_{25} 625 = 2$.

On a slightly more mature note, fixing a base $b > 1$, we may look at logarithm as a function from positive real numbers to all real numbers. This function, called the *logarithmic function*, is defined by

$$\log_b : \mathbf{R}^+ \to \mathbf{R}$$
$$x \to \log_b x = y \quad \text{if } b^y = x$$

As before if the base $b = 10$, we say it is *common logarithms* and if $b = e$, then we say it is *natural logarithms*. Often natural logarithm is denoted by *ln*. *In this chapter, log x denotes the logarithm function to base e*, i.e., *ln x* will be written as simply $\log x$. The Fig 5.10 gives the plots of logarithm function to base 2, e and 10.

Some of the important observations about the logarithm function to any base $b > 1$ are listed below:

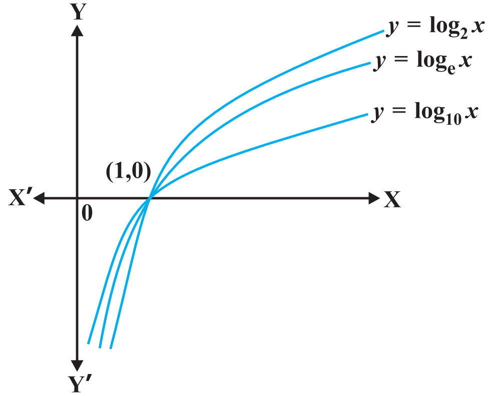


Fig 5.10

(1) We cannot make a meaningful definition of logarithm of non-positive numbers and hence the domain of log function is $\mathbf{R}^+$.

(2) The range of log function is the set of all real numbers.

(3) The point (1, 0) is always on the graph of the log function.

(4) The log function is ever increasing, i.e., as we move from left to right the graph rises above.

(5) For x very near to zero, the value of log x can be made lesser than any given real number. In other words in the fourth quadrant the graph approaches y-axis (but never meets it).

(6) Fig 5.11 gives the plot of $y = e^x$ and $y = ln\, x$. It is of interest to observe that the two curves are the mirror images of each other reflected in the line $y = x$.

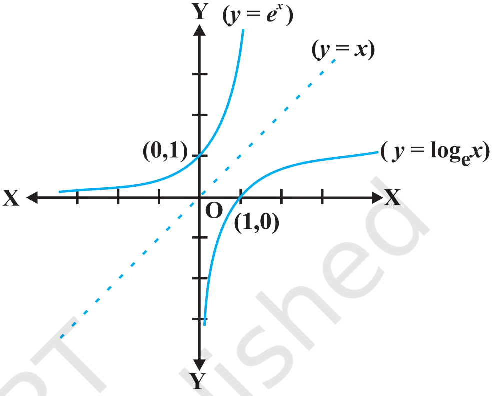


Fig 5.11

Two properties of 'log' functions are proved below:

(1) There is a standard change of base rule to obtain $\log_a p$ in terms of $\log_b p$. Let $\log_a p = \alpha$, $\log_b p = \beta$ and $\log_b a = \gamma$. This means $a^\alpha = p$, $b^\beta = p$ and $b^\gamma = a$.

Substituting the third equation in the first one, we have

$$(b^\gamma)^\alpha = b^{\gamma\alpha} = p$$

Using this in the second equation, we get

$$b^\beta = p = b^{\gamma\alpha}$$

which implies $\beta = \alpha\gamma$ or $\alpha = \dfrac{\beta}{\gamma}$. But then

$$\mathbf{\log_a p = \frac{\log_b p}{\log_b a}}$$

(2) Another interesting property of the log function is its effect on products. Let $\log_b pq = \alpha$. Then $b^\alpha = pq$. If $\log_b p = \beta$ and $\log_b q = \gamma$, then $b^\beta = p$ and $b^\gamma = q$. But then $b^\alpha = pq = b^\beta b^\gamma = b^{\beta + \gamma}$

which implies $\alpha = \beta + \gamma$, i.e.,

$$\mathbf{\log_b pq = \log_b p + \log_b q}$$

A particularly interesting and important consequence of this is when $p = q$. In this case the above may be rewritten as

$$\log_b p^2 = \log_b p + \log_b p = 2 \log p$$

An easy generalisation of this (left as an exercise!) is

$$\log_b p^n = n \log p$$

for any positive integer n. In fact this is true for any real number n, but we will not attempt to prove this. On the similar lines the reader is invited to verify

$$\log_b \frac{x}{y} = \log_b x - \log_b y$$

Example 25 Is it true that $x = e^{\log x}$ for all real x?

Solution First, observe that the domain of log function is set of all positive real numbers. So the above equation is not true for non-positive real numbers. Now, let $y = e^{\log x}$. If $y > 0$, we may take logarithm which gives us $\log y = \log (e^{\log x}) = \log x \cdot \log e = \log x$. Thus $y = x$. Hence $x = e^{\log x}$ is true only for positive values of x.

One of the striking properties of the natural exponential function in differential calculus is that it doesn't change during the process of differentiation. This is captured in the following theorem whose proof we skip.

Theorem 5*

(1) The derivative of e^x w.r.t., x is e^x; i.e., $\frac{d}{dx}(e^x) = e^x$.

(2) The derivative of $\log x$ w.r.t., x is $\frac{1}{x}$; i.e., $\frac{d}{dx}(\log x) = \frac{1}{x}$.

Example 26 Differentiate the following w.r.t. x:

(i) e^{-x} (ii) $\sin (\log x), x > 0$ (iii) $\cos^{-1} (e^x)$ (iv) $e^{\cos x}$

Solution

(i) Let $y = e^{-x}$. Using chain rule, we have

$$\frac{dy}{dx} = e^{-x} \cdot \frac{d}{dx}(-x) = -e^{-x}$$

(ii) Let $y = \sin (\log x)$. Using chain rule, we have

$$\frac{dy}{dx} = \cos (\log x) \cdot \frac{d}{dx}(\log x) = \frac{\cos (\log x)}{x}$$

* *Please see supplementary material on Page 222.*

(iii) Let $y = \cos^{-1}(e^x)$. Using chain rule, we have

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-(e^x)^2}} \cdot \frac{d}{dx}(e^x) = \frac{-e^x}{\sqrt{1-e^{2x}}}$$

(iv) Let $y = e^{\cos x}$. Using chain rule, we have

$$\frac{dy}{dx} = e^{\cos x} \cdot (-\sin x) = -(\sin x)\, e^{\cos x}$$

EXERCISE 5.4

Differentiate the following w.r.t. x:

1. $\dfrac{e^x}{\sin x}$
2. $e^{\sin^{-1} x}$
3. e^{x^3}
4. $\sin(\tan^{-1} e^{-x})$
5. $\log(\cos e^x)$
6. $e^x + e^{x^2} + ... + e^{x^5}$
7. $\sqrt{e^{\sqrt{x}}},\ x > 0$
8. $\log(\log x),\ x > 1$
9. $\dfrac{\cos x}{\log x},\ x > 0$
10. $\cos(\log x + e^x),\ x > 0$

5.5. Logarithmic Differentiation

In this section, we will learn to differentiate certain special class of functions given in the form

$$y = f(x) = [u(x)]^{v(x)}$$

By taking logarithm (to base e) the above may be rewritten as

$$\log y = v(x) \log [u(x)]$$

Using chain rule we may differentiate this to get

$$\frac{1}{y} \cdot \frac{dy}{dx} = v(x) \cdot \frac{1}{u(x)} \cdot u'(x) + v'(x) \cdot \log [u(x)]$$

which implies that

$$\frac{dy}{dx} = y\left[\frac{v(x)}{u(x)} \cdot u'(x) + v'(x) \cdot \log[u(x)]\right]$$

The main point to be noted in this method is that $f(x)$ and $u(x)$ must always be positive as otherwise their logarithms are not defined. This process of differentiation is known as *logarithms differentiation* and is illustrated by the following examples:

Example 27 Differentiate $\sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}$ w.r.t. x.

Solution Let $y=\sqrt{\frac{(x-3)(x^2+4)}{(3x^2+4x+5)}}$

Taking logarithm on both sides, we have

$$\log y = \frac{1}{2}[\log (x-3) + \log (x^2+4) - \log (3x^2+4x+5)]$$

Now, differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{(x-3)}+\frac{2x}{x^2+4}-\frac{6x+4}{3x^2+4x+5}\right]$$

or $$\frac{dy}{dx} = \frac{y}{2}\left[\frac{1}{(x-3)}+\frac{2x}{x^2+4}-\frac{6x+4}{3x^2+4x+5}\right]$$

$$= \frac{1}{2}\sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}\left[\frac{1}{(x-3)}+\frac{2x}{x^2+4}-\frac{6x+4}{3x^2+4x+5}\right]$$

Example 28 Differentiate a^x w.r.t. x, where a is a positive constant.

Solution Let $y = a^x$. Then

$$\log y = x \log a$$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{y}\frac{dy}{dx} = \log a$$

or $$\frac{dy}{dx} = y \log a$$

Thus $$\frac{d}{dx}(a^x) = a^x \log a$$

Alternatively $$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x\log a}) = e^{x\log a}\frac{d}{dx}(x\log a)$$

$$= e^{x\log a} . \log a = a^x \log a.$$

Example 29 Differentiate $x^{\sin x}$, $x > 0$ w.r.t. x.

Solution Let $y = x^{\sin x}$. Taking logarithm on both sides, we have

$$\log y = \sin x \log x$$

Therefore $$\frac{1}{y}\cdot\frac{dy}{dx} = \sin x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(\sin x)$$

or $$\frac{1}{y}\frac{dy}{dx} = (\sin x)\frac{1}{x} + \log x \cos x$$

or $$\frac{dy}{dx} = y\left[\frac{\sin x}{x} + \cos x \log x\right]$$

$$= x^{\sin x}\left[\frac{\sin x}{x} + \cos x \log x\right]$$

$$= x^{\sin x - 1}\cdot\sin x + x^{\sin x}\cdot\cos x \log x$$

Example 30 Find $\frac{dy}{dx}$, if $y^x + x^y + x^x = a^b$.

Solution Given that $y^x + x^y + x^x = a^b$.

Putting $u = y^x$, $v = x^y$ and $w = x^x$, we get $u + v + w = a^b$

Therefore $$\frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} = 0 \quad \text{... (1)}$$

Now, $u = y^x$. Taking logarithm on both sides, we have

$$\log u = x \log y$$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{u}\cdot\frac{du}{dx} = x\frac{d}{dx}(\log y) + \log y \frac{d}{dx}(x)$$

$$= x\frac{1}{y}\cdot\frac{dy}{dx} + \log y \cdot 1$$

So $$\frac{du}{dx} = u\left(\frac{x}{y}\frac{dy}{dx} + \log y\right) = y^x\left[\frac{x}{y}\frac{dy}{dx} + \log y\right] \quad \text{... (2)}$$

Also $v = x^y$

Taking logarithm on both sides, we have

$$\log v = y \log x$$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{v}\cdot\frac{dv}{dx} = y\frac{d}{dx}(\log x) + \log x\frac{dy}{dx}$$

$$= y\cdot\frac{1}{x} + \log x\cdot\frac{dy}{dx}$$

So $$\frac{dv}{dx} = v\left[\frac{y}{x} + \log x\frac{dy}{dx}\right]$$

$$= x^y\left[\frac{y}{x} + \log x\frac{dy}{dx}\right] \qquad \text{... (3)}$$

Again $$w = x^x$$

Taking logarithm on both sides, we have

$$\log w = x \log x.$$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{w}\cdot\frac{dw}{dx} = x\frac{d}{dx}(\log x) + \log x\cdot\frac{d}{dx}(x)$$

$$= x\cdot\frac{1}{x} + \log x\cdot 1$$

i.e. $$\frac{dw}{dx} = w\,(1 + \log x)$$

$$= x^x\,(1 + \log x) \qquad \text{... (4)}$$

From (1), (2), (3), (4), we have

$$y^x\left(\frac{x}{y}\frac{dy}{dx} + \log y\right) + x^y\left(\frac{y}{x} + \log x\frac{dy}{dx}\right) + x^x\,(1 + \log x) = 0$$

or $$(x\,.\,y^{x-1} + x^y\,.\,\log x)\,\frac{dy}{dx} = -\,x^x\,(1 + \log x) - y\,.\,x^{y-1} - y^x \log y$$

Therefore $$\frac{dy}{dx} = \frac{-[y^x \log y + y\,.\,x^{y-1} + x^x(1+\log x)]}{x\,.\,y^{x-1} + x^y \log x}$$

EXERCISE 5.5

Differentiate the functions given in Exercises 1 to 11 w.r.t. x.

1. $\cos x \cdot \cos 2x \cdot \cos 3x$
2. $\sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$
3. $(\log x)^{\cos x}$
4. $x^x - 2^{\sin x}$
5. $(x+3)^2 \cdot (x+4)^3 \cdot (x+5)^4$
6. $\left(x+\dfrac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$
7. $(\log x)^x + x^{\log x}$
8. $(\sin x)^x + \sin^{-1}\sqrt{x}$
9. $x^{\sin x} + (\sin x)^{\cos x}$
10. $x^{x\cos x} + \dfrac{x^2+1}{x^2-1}$
11. $(x\cos x)^x + (x\sin x)^{\frac{1}{x}}$

Find $\dfrac{dy}{dx}$ of the functions given in Exercises 12 to 15.

12. $x^y + y^x = 1$
13. $y^x = x^y$
14. $(\cos x)^y = (\cos y)^x$
15. $xy = e^{(x-y)}$
16. Find the derivative of the function given by $f(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)$ and hence find $f'(1)$.
17. Differentiate $(x^2 - 5x + 8)(x^3 + 7x + 9)$ in three ways mentioned below:
    (i) by using product rule
    (ii) by expanding the product to obtain a single polynomial.
    (iii) by logarithmic differentiation.
    Do they all give the same answer?
18. If u, v and w are functions of x, then show that
$$\frac{d}{dx}(u.\,v.\,w) = \frac{du}{dx}\,v.\,w + u\,.\,\frac{dv}{dx}\,.\,w + u\,.\,v\,\frac{dw}{dx}$$
    in two ways - first by repeated application of product rule, second by logarithmic differentiation.

5.6 Derivatives of Functions in Parametric Forms

Sometimes the relation between two variables is neither explicit nor implicit, but some link of a third variable with each of the two variables, separately, establishes a relation between the first two variables. In such a situation, we say that the relation between them is expressed via a third variable. The third variable is called the parameter. More precisely, a relation expressed between two variables x and y in the form $x = f(t)$, $y = g(t)$ is said to be parametric form with t as a parameter.

In order to find derivative of function in such form, we have by chain rule.

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

or
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \left(\text{whenever } \frac{dx}{dt} \neq 0\right)$$

Thus
$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)} \quad \left(\text{as } \frac{dy}{dt} = g'(t) \text{ and } \frac{dx}{dt} = f'(t)\right) \quad [\text{provided } f'(t) \neq 0]$$

Example 31 Find $\frac{dy}{dx}$, if $x = a \cos \theta$, $y = a \sin \theta$.

Solution Given that

$$x = a \cos \theta,\ y = a \sin \theta$$

Therefore
$$\frac{dx}{d\theta} = -a \sin \theta, \quad \frac{dy}{d\theta} = a \cos \theta$$

Hence
$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a \cos \theta}{-a \sin \theta} = -\cot \theta$$

Example 32 Find $\frac{dy}{dx}$, if $x = at^2$, $y = 2at$.

Solution Given that $x = at^2$, $y = 2at$

So
$$\frac{dx}{dt} = 2at \quad \text{and} \quad \frac{dy}{dt} = 2a$$

Therefore
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

Example 33 Find $\frac{dy}{dx}$, if $x = a\ (\theta + \sin\theta)$, $y = a\ (1 - \cos\theta)$.

Solution We have $\frac{dx}{d\theta} = a(1 + \cos\theta)$, $\frac{dy}{d\theta} = a\ (\sin\theta)$

Therefore
$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1+\cos\theta)} = \tan\frac{\theta}{2}$$

> **Note** It may be noted here that $\frac{dy}{dx}$ is expressed in terms of parameter only without directly involving the main variables x and y.

Example 34 Find $\frac{dy}{dx}$, if $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Solution Let $x = a\cos^3\theta$, $y = a\sin^3\theta$. Then

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = (a\cos^3\theta)^{\frac{2}{3}} + (a\sin^3\theta)^{\frac{2}{3}}$$
$$= a^{\frac{2}{3}}(\cos^2\theta + (\sin^2\theta) = a^{\frac{2}{3}}$$

Hence, $x = a\cos^3\theta$, $y = a\sin^3\theta$ is parametric equation of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

Now
$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta \text{ and } \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

Therefore
$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta = -\sqrt[3]{\frac{y}{x}}$$

EXERCISE 5.6

If x and y are connected parametrically by the equations given in Exercises 1 to 10, without eliminating the parameter, Find $\frac{dy}{dx}$.

1. $x = 2at^2,\ y = at^4$
2. $x = a\cos\theta,\ y = b\cos\theta$
3. $x = \sin t,\ y = \cos 2t$
4. $x = 4t,\ y = \frac{4}{t}$
5. $x = \cos\theta - \cos 2\theta,\ y = \sin\theta - \sin 2\theta$
6. $x = a(\theta - \sin\theta),\ y = a(1 + \cos\theta)$
7. $x = \frac{\sin^3 t}{\sqrt{\cos 2t}},\ y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$
8. $x = a\left(\cos t + \log\tan\frac{t}{2}\right)\ y = a\sin t$
9. $x = a\sec\theta,\ y = b\tan\theta$
10. $x = a(\cos\theta + \theta\sin\theta),\ y = a(\sin\theta - \theta\cos\theta)$
11. If $x = \sqrt{a^{\sin^{-1}t}},\ y = \sqrt{a^{\cos^{-1}t}}$, show that $\frac{dy}{dx} = -\frac{y}{x}$

5.7 Second Order Derivative

Let $y = f(x)$. Then

$$\frac{dy}{dx} = f'(x) \qquad \text{... (1)}$$

If $f'(x)$ is differentiable, we may differentiate (1) again w.r.t. x. Then, the left hand side becomes $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ which is called the *second order derivative* of y w.r.t. x and is denoted by $\frac{d^2y}{dx^2}$. The second order derivative of $f(x)$ is denoted by $f''(x)$. It is also denoted by D^2y or y'' or y_2 if $y = f(x)$. We remark that higher order derivatives may be defined similarly.

Example 35 Find $\frac{d^2y}{dx^2}$, if $y = x^3 + \tan x$.

Solution Given that $y = x^3 + \tan x$. Then

$$\frac{dy}{dx} = 3x^2 + \sec^2 x$$

Therefore $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(3x^2 + \sec^2 x\right)$$

$$= 6x + 2 \sec x \,.\, \sec x \tan x = 6x + 2 \sec^2 x \tan x$$

Example 36 If $y = \text{A} \sin x + \text{B} \cos x$, then prove that $\frac{d^2y}{dx^2} + y = 0$.

Solution We have

$$\frac{dy}{dx} = \text{A} \cos x - \text{B} \sin x$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}(\text{A} \cos x - \text{B} \sin x)$$

$$= -\text{A} \sin x - \text{B} \cos x = -y$$

Hence $$\frac{d^2y}{dx^2} + y = 0$$

Example 37 If $y = 3e^{2x} + 2e^{3x}$, prove that $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$.

Solution Given that $y = 3e^{2x} + 2e^{3x}$. Then

$$\frac{dy}{dx} = 6e^{2x} + 6e^{3x} = 6\,(e^{2x} + e^{3x})$$

Therefore $$\frac{d^2y}{dx^2} = 12e^{2x} + 18e^{3x} = 6\,(2e^{2x} + 3e^{3x})$$

Hence $$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 6\,(2e^{2x} + 3e^{3x})$$

$$- 30\,(e^{2x} + e^{3x}) + 6\,(3e^{2x} + 2e^{3x}) = 0$$

Example 38 If $y = \sin^{-1} x$, show that $(1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 0$.

Solution We have $y = \sin^{-1}x$. Then

$$\frac{dy}{dx} = \frac{1}{\sqrt{(1-x^2)}}$$

or
$$\sqrt{(1-x^2)}\,\frac{dy}{dx} = 1$$

So
$$\frac{d}{dx}\left(\sqrt{(1-x^2)}\,.\,\frac{dy}{dx}\right) = 0$$

or
$$\sqrt{(1-x^2)}\cdot\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot\frac{d}{dx}\left(\sqrt{(1-x^2)}\right) = 0$$

or
$$\sqrt{(1-x^2)}\cdot\frac{d^2y}{dx^2} - \frac{dy}{dx}\cdot\frac{2x}{2\sqrt{1-x^2}} = 0$$

Hence
$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$$

Alternatively, Given that $y = \sin^{-1} x$, we have

$$y_1 = \frac{1}{\sqrt{1-x^2}}, \text{ i.e., } \left(1-x^2\right)y_1^2 = 1$$

So
$$(1-x^2)\,.\,2y_1y_2 + y_1^2(0-2x) = 0$$

Hence
$$(1 - x^2)\, y_2 - xy_1 = 0$$

EXERCISE 5.7

Find the second order derivatives of the functions given in Exercises 1 to 10.

1. $x^2 + 3x + 2$ **2.** x^{20} **3.** $x\ .\ \cos x$

4. $\log x$ **5.** $x^3 \log x$ **6.** $e^x \sin 5x$

7. $e^{6x} \cos 3x$ **8.** $\tan^{-1} x$ **9.** $\log (\log x)$

10. $\sin (\log x)$

11. If $y = 5 \cos x - 3 \sin x$, prove that $\dfrac{d^2y}{dx^2} + y = 0$

12. If $y = \cos^{-1} x$, Find $\frac{d^2y}{dx^2}$ in terms of y alone.

13. If $y = 3 \cos (\log x) + 4 \sin (\log x)$, show that $x^2 y_2 + xy_1 + y = 0$

14. If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$

15. If $y = 500e^{7x} + 600e^{-7x}$, show that $\frac{d^2y}{dx^2} = 49y$

16. If $e^y (x + 1) = 1$, show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

17. If $y = (\tan^{-1} x)^2$, show that $(x^2 + 1)^2 y_2 + 2x (x^2 + 1) y_1 = 2$

Miscellaneous Examples

Example 39 Differentiate w.r.t. x, the following function:

(i) $\sqrt{3x+2} + \frac{1}{\sqrt{2x^2+4}}$ (ii) $\log_7 (\log x)$

Solution

(i) Let $y = \sqrt{3x+2} + \frac{1}{\sqrt{2x^2+4}} = (3x+2)^{\frac{1}{2}} + (2x^2+4)^{-\frac{1}{2}}$

Note that this function is defined at all real numbers $x > -\frac{2}{3}$. Therefore

$$\frac{dy}{dx} = \frac{1}{2}(3x+2)^{\frac{1}{2}-1} \cdot \frac{d}{dx}(3x+2) + \left(-\frac{1}{2}\right)(2x^2+4)^{-\frac{1}{2}-1} \cdot \frac{d}{dx}(2x^2+4)$$

$$= \frac{1}{2}(3x+2)^{-\frac{1}{2}} \cdot (3) - \left(\frac{1}{2}\right)(2x^2+4)^{-\frac{3}{2}} \cdot 4x$$

$$= \frac{3}{2\sqrt{3x+2}} - \frac{2x}{\left(2x^2+4\right)^{\frac{3}{2}}}$$

This is defined for all real numbers $x > -\frac{2}{3}$.

(ii) Let $y = \log_7 (\log x) = \dfrac{\log (\log x)}{\log 7}$ (by change of base formula).

The function is defined for all real numbers $x > 1$. Therefore

$$\frac{dy}{dx} = \frac{1}{\log 7} \frac{d}{dx}(\log (\log x))$$

$$= \frac{1}{\log 7} \frac{1}{\log x} \cdot \frac{d}{dx}(\log x)$$

$$= \frac{1}{x \log 7 \log x}$$

Example 40 Differentiate the following w.r.t. x.

(i) $\cos^{-1} (\sin x)$ (ii) $\tan^{-1}\left(\dfrac{\sin x}{1+\cos x}\right)$ (iii) $\sin^{-1}\left(\dfrac{2^{x+1}}{1+4^x}\right)$

Solution

(i) Let $f(x) = \cos^{-1} (\sin x)$. Observe that this function is defined for all real numbers. We may rewrite this function as

$$f(x) = \cos^{-1} (\sin x)$$

$$= \cos^{-1}\left[\cos\left(\frac{\pi}{2} - x\right)\right]$$

$$= \frac{\pi}{2} - x$$

Thus $f'(x) = -1$.

(ii) Let $f(x) = \tan^{-1}\left(\dfrac{\sin x}{1+\cos x}\right)$. Observe that this function is defined for all real numbers, where $\cos x \neq -1$; i.e., at all odd multiples of π. We may rewrite this function as

$$f(x) = \tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$$

$$= \tan^{-1}\left[\frac{2 \sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{2\cos^2 \frac{x}{2}}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{x}{2}\right)\right] = \frac{x}{2}$$

Observe that we could cancel $\cos\left(\frac{x}{2}\right)$ in both numerator and denominator as it is not equal to zero. Thus $f'(x) = \frac{1}{2}\cdot$

(iii) Let $f(x) = \sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$. To find the domain of this function we need to find all x such that $-1 \le \frac{2^{x+1}}{1+4^x} \le 1$. Since the quantity in the middle is always positive, we need to find all x such that $\frac{2^{x+1}}{1+4^x} \le 1$, i.e., all x such that $2^{x+1} \le 1 + 4^x$. We may rewrite this as $2 \le \frac{1}{2^x} + 2^x$ which is true for all x. Hence the function is defined at every real number. By putting $2^x = \tan\theta$, this function may be rewritten as

$$f(x) = \sin^{-1}\left[\frac{2^{x+1}}{1+4^x}\right]$$

$$= \sin^{-1}\left[\frac{2^x \cdot 2}{1+(2^x)^2}\right]$$

$$= \sin^{-1}\left[\frac{2\tan\theta}{1+\tan^2\theta}\right]$$

$$= \sin^{-1}[\sin 2\theta]$$

$$= 2\theta = 2\tan^{-1}(2^x)$$

Thus $$f'(x) = 2\cdot\frac{1}{1+(2^x)^2}\cdot\frac{d}{dx}(2^x)$$

$$= \frac{2}{1+4^x}\cdot(2^x)\log 2$$

$$= \frac{2^{x+1}\log 2}{1+4^x}$$

Example 41 Find $f'(x)$ if $f(x) = (\sin x)^{\sin x}$ for all $0 < x < \pi$.

Solution The function $y = (\sin x)^{\sin x}$ is defined for all positive real numbers. Taking logarithms, we have

$$\log y = \log (\sin x)^{\sin x} = \sin x \log (\sin x)$$

Then
$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(\sin x \log (\sin x))$$

$$= \cos x \log (\sin x) + \sin x \,.\, \frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x)$$

$$= \cos x \log (\sin x) + \cos x$$

$$= (1 + \log (\sin x)) \cos x$$

Thus
$$\frac{dy}{dx} = y((1 + \log (\sin x)) \cos x) = (1 + \log (\sin x)) (\sin x)^{\sin x} \cos x$$

Example 42 For a positive constant a find $\frac{dy}{dx}$, where

$$y = a^{t+\frac{1}{t}}, \text{ and } x = \left(t+\frac{1}{t}\right)^a$$

Solution Observe that both y and x are defined for all real $t \neq 0$. Clearly

$$\frac{dy}{dt} = \frac{d}{dt}\left(a^{t+\frac{1}{t}}\right) = a^{t+\frac{1}{t}}\frac{d}{dt}\left(t+\frac{1}{t}\right)\cdot\log a$$

$$= a^{t+\frac{1}{t}}\left(1-\frac{1}{t^2}\right)\log a$$

Similarly
$$\frac{dx}{dt} = a\left[t+\frac{1}{t}\right]^{a-1}\cdot\frac{d}{dt}\left(t+\frac{1}{t}\right)$$

$$= a\left[t+\frac{1}{t}\right]^{a-1}\cdot\left(1-\frac{1}{t^2}\right)$$

$\frac{dx}{dt} \neq 0$ only if $t \neq \pm 1$. Thus for $t \neq \pm 1$,

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{a^{t+\frac{1}{t}}\left(1-\frac{1}{t^2}\right)\log a}{a\left[t+\frac{1}{t}\right]^{a-1}\cdot\left(1-\frac{1}{t^2}\right)}$$

$$=\frac{a^{t+\frac{1}{t}}\log a}{a\left(t+\frac{1}{t}\right)^{a-1}}$$

Example 43 Differentiate $\sin^2 x$ w.r.t. $e^{\cos x}$.

Solution Let $u(x) = \sin^2 x$ and $v(x) = e^{\cos x}$. We want to find $\frac{du}{dv}=\frac{du/dx}{dv/dx}$. Clearly

$$\frac{du}{dx} = 2\sin x\cos x \text{ and } \frac{dv}{dx} = e^{\cos x}(-\sin x) = -(\sin x)\, e^{\cos x}$$

Thus

$$\frac{du}{dv}=\frac{2\sin x\cos x}{-\sin x\, e^{\cos x}}=-\frac{2\cos x}{e^{\cos x}}$$

Miscellaneous Exercise on Chapter 5

Differentiate w.r.t. x the function in Exercises 1 to 11.

1. $(3x^2 - 9x + 5)^9$

2. $\sin^3 x + \cos^6 x$

3. $(5x)^{3\cos 2x}$

4. $\sin^{-1}(x\sqrt{x}),\ 0 \le x \le 1$

5. $\dfrac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}},\ -2 < x < 2$

6. $\cot^{-1}\left[\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right],\ 0 < x < \dfrac{\pi}{2}$

7. $(\log x)^{\log x},\ x > 1$

8. $\cos(a\cos x + b\sin x)$, for some constant a and b.

9. $(\sin x - \cos x)^{(\sin x - \cos x)},\ \dfrac{\pi}{4} < x < \dfrac{3\pi}{4}$

10. $x^x + x^a + a^x + a^a$, for some fixed $a > 0$ and $x > 0$

11. $x^{x^2-3} + (x-3)^{x^2}$, for $x > 3$

12. Find $\frac{dy}{dx}$, if $y = 12(1 - \cos t)$, $x = 10(t - \sin t)$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$

13. Find $\frac{dy}{dx}$, if $y = \sin^{-1} x + \sin^{-1} \sqrt{1-x^2}$, $0 < x < 1$

14. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, for , $-1 < x < 1$, prove that

$$\frac{dy}{dx} = -\frac{1}{(1+x)^2}$$

15. If $(x - a)^2 + (y - b)^2 = c^2$, for some $c > 0$, prove that

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

is a constant independent of a and b.

16. If $\cos y = x \cos(a + y)$, with $\cos a \neq \pm 1$, prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$.

17. If $x = a(\cos t + t \sin t)$ and $y = a(\sin t - t \cos t)$, find $\frac{d^2y}{dx^2}$.

18. If $f(x) = |x|^3$, show that $f''(x)$ exists for all real x and find it.

19. Using the fact that sin (A + B) = sin A cos B + cos A sin B and the differentiation, obtain the sum formula for cosines.

20. Does there exist a function which is continuous everywhere but not differentiable at exactly two points? Justify your answer.

21. If $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$, prove that $\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

22. If $y = e^{a\cos^{-1}x}$, $-1 \leq x \leq 1$, show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$.

Summary

- A real valued function is ***continuous*** at a point in its domain if the limit of the function at that point equals the value of the function at that point. A function is continuous if it is continuous on the whole of its domain.
- Sum, difference, product and quotient of continuous functions are continuous. i.e., if f and g are continuous functions, then

  $(f \pm g)(x) = f(x) \pm g(x)$ is continuous.

  $(f \,.\, g)(x) = f(x) \,.\, g(x)$ is continuous.

  $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ (wherever $g(x) \neq 0$) is continuous.
- Every differentiable function is continuous, but the converse is not true.
- Chain rule is rule to differentiate composites of functions. If $f = v \text{ o } u$, $t = u(x)$ and if both $\frac{dt}{dx}$ and $\frac{dv}{dt}$ exist then

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$$

- Following are some of the standard derivatives (in appropriate domains):

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(e^x\right) = e^x \qquad \frac{d}{dx}\left(\log x\right) = \frac{1}{x}$$

- Logarithmic differentiation is a powerful technique to differentiate functions of the form $f(x) = [u(x)]^{v(x)}$. Here both $f(x)$ and $u(x)$ need to be positive for this technique to make sense.

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Chapter 6

APPLICATION OF DERIVATIVES

❖ *With the Calculus as a key, Mathematics can be successfully applied to the explanation of the course of Nature." — WHITEHEAD* ❖

6.1 Introduction

In Chapter 5, we have learnt how to find derivative of composite functions, inverse trigonometric functions, implicit functions, exponential functions and logarithmic functions. In this chapter, we will study applications of the derivative in various disciplines, e.g., in engineering, science, social science, and many other fields. For instance, we will learn how the derivative can be used (i) to determine rate of change of quantities, (ii) to find the equations of tangent and normal to a curve at a point, (iii) to find turning points on the graph of a function which in turn will help us to locate points at which largest or smallest value (locally) of a function occurs. We will also use derivative to find intervals on which a function is increasing or decreasing. Finally, we use the derivative to find approximate value of certain quantities.

6.2 Rate of Change of Quantities

Recall that by the derivative $\frac{ds}{dt}$, we mean the rate of change of distance s with respect to the time t. In a similar fashion, whenever one quantity y varies with another quantity x, satisfying some rule $y = f(x)$, then $\frac{dy}{dx}$ (or $f'(x)$) represents the rate of change of y with respect to x and $\left.\frac{dy}{dx}\right]_{x=x_0}$ (or $f'(x_0)$) represents the rate of change of y with respect to x at $x = x_0$.

Further, if two variables x and y are varying with respect to another variable t, i.e., if $x = f(t)$ and $y = g(t)$, then by Chain Rule

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}, \text{ if } \frac{dx}{dt} \neq 0$$

Thus, the rate of change of y with respect to x can be calculated using the rate of change of y and that of x both with respect to t.

Let us consider some examples.

Example 1 Find the rate of change of the area of a circle per second with respect to its radius r when $r = 5$ cm.

Solution The area A of a circle with radius r is given by $A = \pi r^2$. Therefore, the rate of change of the area A with respect to its radius r is given by $\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r$.

When $r = 5$ cm, $\frac{dA}{dr} = 10\pi$. Thus, the area of the circle is changing at the rate of 10π cm²/s.

Example 2 The volume of a cube is increasing at a rate of 9 cubic centimetres per second. How fast is the surface area increasing when the length of an edge is 10 centimetres ?

Solution Let x be the length of a side, V be the volume and S be the surface area of the cube. Then, $V = x^3$ and $S = 6x^2$, where x is a function of time t.

Now $$\frac{dV}{dt} = 9\text{cm}^3/\text{s (Given)}$$

Therefore $$9 = \frac{dV}{dt} = \frac{d}{dt}(x^3) = \frac{d}{dx}(x^3)\cdot\frac{dx}{dt} \quad \text{(By Chain Rule)}$$

$$= 3x^2 \cdot \frac{dx}{dt}$$

or $$\frac{dx}{dt} = \frac{3}{x^2} \quad \text{... (1)}$$

Now $$\frac{dS}{dt} = \frac{d}{dt}(6x^2) = \frac{d}{dx}(6x^2)\cdot\frac{dx}{dt} \quad \text{(By Chain Rule)}$$

$$= 12x\cdot\left(\frac{3}{x^2}\right) = \frac{36}{x} \quad \text{(Using (1))}$$

Hence, when $x = 10$ cm, $\frac{dS}{dt} = 3.6\ \text{cm}^2/\text{s}$

Example 3 A stone is dropped into a quiet lake and waves move in circles at a speed of 4cm per second. At the instant, when the radius of the circular wave is 10 cm, how fast is the enclosed area increasing?

Solution The area A of a circle with radius r is given by $A = \pi r^2$. Therefore, the rate of change of area A with respect to time t is

$$\frac{dA}{dt} = \frac{d}{dt}(\pi r^2) = \frac{d}{dr}(\pi r^2) \cdot \frac{dr}{dt} = 2\pi r \frac{dr}{dt} \quad \text{(By Chain Rule)}$$

It is given that $\frac{dr}{dt} = 4\text{cm/s}$

Therefore, when $r = 10$ cm, $\frac{dA}{dt} = 2\pi(10)(4) = 80\pi$

Thus, the enclosed area is increasing at the rate of 80π cm^2/s, when $r = 10$ cm.

> **Note** $\frac{dy}{dx}$ is positive if y increases as x increases and is negative if y decreases as x increases.

Example 4 The length x of a rectangle is decreasing at the rate of 3 cm/minute and the width y is increasing at the rate of 2cm/minute. When x =10cm and y = 6cm, find the rates of change of (a) the perimeter and (b) the area of the rectangle.

Solution Since the length x is decreasing and the width y is increasing with respect to time, we have

$$\frac{dx}{dt} = -3 \text{ cm/min} \quad \text{and} \quad \frac{dy}{dt} = 2 \text{ cm/min}$$

(a) The perimeter P of a rectangle is given by

$$P = 2(x + y)$$

Therefore $$\frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = 2(-3+2) = -2 \text{ cm/min}$$

(b) The area A of the rectangle is given by

$$A = x \cdot y$$

Therefore $$\frac{dA}{dt} = \frac{dx}{dt} \cdot y + x \cdot \frac{dy}{dt}$$

$$= -3(6) + 10(2) \quad (\text{as } x = 10 \text{ cm and } y = 6 \text{ cm})$$

$$= 2 \text{ cm}^2/\text{min}$$

Example 5 The total cost C(x) in Rupees, associated with the production of x units of an item is given by

$$C(x) = 0.005\, x^3 - 0.02\, x^2 + 30x + 5000$$

Find the marginal cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of change of total cost at any level of output.

Solution Since marginal cost is the rate of change of total cost with respect to the output, we have

Marginal cost (MC) $= \frac{dC}{dx} = 0.005(3x^2) - 0.02(2x) + 30$

When $x = 3$, MC $= 0.015(3^2) - 0.04(3) + 30$

$= 0.135 - 0.12 + 30 = 30.015$

Hence, the required marginal cost is ₹ 30.02 (nearly).

Example 6 The total revenue in Rupees received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$. Find the marginal revenue, when $x = 5$, where by marginal revenue we mean the rate of change of total revenue with respect to the number of items sold at an instant.

Solution Since marginal revenue is the rate of change of total revenue with respect to the number of units sold, we have

Marginal Revenue (MR) $= \frac{dR}{dx} = 6x + 36$

When $x = 5$, MR $= 6(5) + 36 = 66$

Hence, the required marginal revenue is ₹ 66.

EXERCISE 6.1

1. Find the rate of change of the area of a circle with respect to its radius r when
   (a) $r = 3$ cm (b) $r = 4$ cm
2. The volume of a cube is increasing at the rate of 8 cm^3/s. How fast is the surface area increasing when the length of an edge is 12 cm?
3. The radius of a circle is increasing uniformly at the rate of 3 cm/s. Find the rate at which the area of the circle is increasing when the radius is 10 cm.
4. An edge of a variable cube is increasing at the rate of 3 cm/s. How fast is the volume of the cube increasing when the edge is 10 cm long?
5. A stone is dropped into a quiet lake and waves move in circles at the speed of 5 cm/s. At the instant when the radius of the circular wave is 8 cm, how fast is the enclosed area increasing?

6. The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increase of its circumference?

7. The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute. When $x = 8$cm and $y = 6$cm, find the rates of change of (a) the perimeter, and (b) the area of the rectangle.

8. A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm.

9. A balloon, which always remains spherical has a variable radius. Find the rate at which its volume is increasing with the radius when the later is 10 cm.

10. A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall ?

11. A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y-coordinate is changing 8 times as fast as the x-coordinate.

12. The radius of an air bubble is increasing at the rate of $\frac{1}{2}$ cm/s. At what rate is the volume of the bubble increasing when the radius is 1 cm?

13. A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}(2x+1)$. Find the rate of change of its volume with respect to x.

14. Sand is pouring from a pipe at the rate of 12 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?

15. The total cost C (x) in Rupees associated with the production of x units of an item is given by

$$C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000.$$

Find the marginal cost when 17 units are produced.

16. The total revenue in Rupees received from the sale of x units of a product is given by

$$R(x) = 13x^2 + 26x + 15.$$

Find the marginal revenue when $x = 7$.

Choose the correct answer for questions 17 and 18.

17. The rate of change of the area of a circle with respect to its radius r at $r = 6$ cm is

(A) 10π (B) 12π (C) 8π (D) 11π

18. The total revenue in Rupees received from the sale of x units of a product is given by
$R(x) = 3x^2 + 36x + 5$. The marginal revenue, when $x = 15$ is
(A) 116 (B) 96 (C) 90 (D) 126

6.3 Increasing and Decreasing Functions

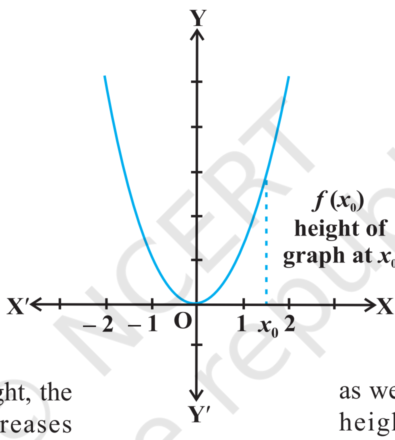
In this section, we will use differentiation to find out whether a function is increasing or decreasing or none.

Consider the function f given by $f(x) = x^2$, $x \in \mathbf{R}$. The graph of this function is a parabola as given in Fig 6.1.

Values left to origin

x	$f(x) = x^2$
-2	4
$-\frac{3}{2}$	$\frac{9}{4}$
-1	1
$-\frac{1}{2}$	$\frac{1}{4}$
0	0

as we move from left to right, the height of the graph decreases



Values right to origin

x	$f(x) = x^2$
0	0
$\frac{1}{2}$	$\frac{1}{4}$
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4

as we move from left to right, the height of the graph increases

Fig 6.1

First consider the graph (Fig 6.1) to the right of the origin. Observe that as we move from left to right along the graph, the height of the graph continuously increases. For this reason, the function is said to be increasing for the real numbers $x > 0$.

Now consider the graph to the left of the origin and observe here that as we move from left to right along the graph, the height of the graph continuously decreases. Consequently, the function is said to be decreasing for the real numbers $x < 0$.

We shall now give the following analytical definitions for a function which is increasing or decreasing on an interval.

Definition 1 Let I be an interval contained in the domain of a real valued function f. Then f is said to be

(i) increasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in$ I.

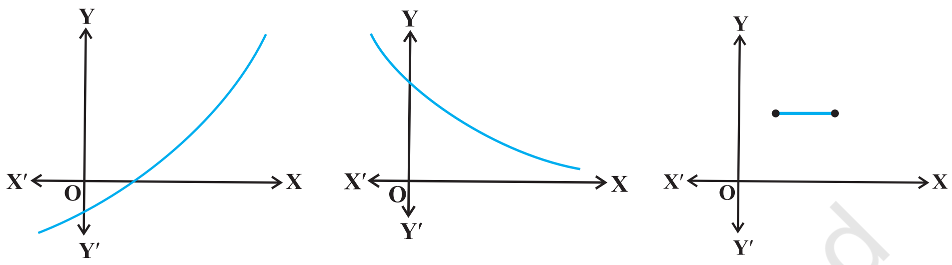
(ii) decreasing on I, if x_1, x_2 in I $\Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in$ I.

(iii) constant on I, if $f(x) = c$ for all $x \in$ I, where c is a constant.

(iv) decreasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in$ I.

(v) strictly decreasing on I if $x_1 < x_2$ in I $\Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in$ I.

For graphical representation of such functions see Fig 6.2.



Strictly Increasing function (i) **Strictly Decreasing function (ii)** **Neither Increasing nor Decreasing function (iii)**

Fig 6.2

We shall now define when a function is increasing or decreasing at a point.

Definition 2 Let x_0 be a point in the domain of definition of a real valued function f. Then f is said to be increasing, decreasing at x_0 if there exists an open interval I containing x_0 such that f is increasing, decreasing, respectively, in I.

Let us clarify this definition for the case of increasing function.

Example 7 Show that the function given by $f(x) = 7x - 3$ is increasing on **R**.

Solution Let x_1 and x_2 be any two numbers in **R**. Then

$$x_1 < x_2 \Rightarrow 7x_1 < 7x_2 \Rightarrow 7x_1 - 3 < 7x_2 - 3 \Rightarrow f(x_1) < f(x_2)$$

Thus, by Definition 1, it follows that f is strictly increasing on **R**.

We shall now give the first derivative test for increasing and decreasing functions. The proof of this test requires the Mean Value Theorem studied in Chapter 5.

Theorem 1 Let f be continuous on $[a, b]$ and differentiable on the open interval (a,b). Then

(a) f is increasing in $[a,b]$ if $f'(x) > 0$ for each $x \in (a, b)$

(b) f is decreasing in $[a,b]$ if $f'(x) < 0$ for each $x \in (a, b)$

(c) f is a constant function in $[a,b]$ if $f'(x) = 0$ for each $x \in (a, b)$

Proof (a) Let $x_1, x_2 \in [a, b]$ be such that $x_1 < x_2$.

Then, by Mean Value Theorem (Theorem 8 in Chapter 5), there exists a point c between x_1 and x_2 such that

$$f(x_2) - f(x_1) = f'(c)\,(x_2 - x_1)$$

i.e. $$f(x_2) - f(x_1) > 0 \qquad (\text{as } f'(c) > 0 \text{ (given)})$$

i.e. $$f(x_2) > f(x_1)$$

Thus, we have

$$x_1 < x_2 \quad f(x_1) \quad f(x_2), \text{ for all } x_1, x_2 \quad [a,b]$$

Hence, f is an increasing function in $[a,b]$.

The proofs of part (b) and (c) are similar. It is left as an exercise to the reader.

Remarks

There is a more generalised theorem, which states that if f¢(x) > 0 for x in an interval excluding the end points and f is continuous in the interval, then f is increasing. Similarly, if f¢(x) < 0 for x in an interval excluding the end points and f is continuous in the interval, then f is decreasing.

Example 8 Show that the function f given by

$$f(x) = x^3 - 3x^2 + 4x,\ x \in \mathbf{R}$$

is increasing on **R**.

Solution Note that

$$\begin{aligned} f'(x) &= 3x^2 - 6x + 4 \\ &= 3(x^2 - 2x + 1) + 1 \\ &= 3(x - 1)^2 + 1 > 0, \text{ in every interval of } \mathbf{R} \end{aligned}$$

Therefore, the function f is increasing on **R**.

Example 9 Prove that the function given by $f(x) = \cos x$ is

(a) decreasing in $(0, \pi)$

(b) increasing in $(\pi, 2\pi)$, and

(c) neither increasing nor decreasing in $(0, 2\pi)$.

Solution Note that $f'(x) = -\sin x$

(a) Since for each $x \in (0, \pi)$, $\sin x > 0$, we have $f'(x) < 0$ and so f is decreasing in $(0, \pi)$.

(b) Since for each $x \in (\pi, 2\pi)$, $\sin x < 0$, we have $f'(x) > 0$ and so f is increasing in $(\pi, 2\pi)$.

(c) Clearly by (a) and (b) above, f is neither increasing nor decreasing in $(0, 2\pi)$.

Example 10 Find the intervals in which the function f given by $f(x) = x^2 - 4x + 6$ is

(a) increasing (b) decreasing

Solution We have

$$f(x) = x^2 - 4x + 6$$

or $$f'(x) = 2x - 4$$

Fig 6.3

Therefore, $f'(x) = 0$ gives $x = 2$. Now the point $x = 2$ divides the real line into two disjoint intervals namely, $(-\infty, 2)$ and $(2, \infty)$ (Fig 6.3). In the interval $(-\infty, 2)$, $f'(x) = 2x - 4 < 0$.

Therefore, f is decreasing in this interval. Also, in the interval $(2, \infty)$, $f'(x) > 0$ and so the function f is increasing in this interval.

Example 11 Find the intervals in which the function f given by $f(x) = 4x^3 - 6x^2 - 72x + 30$ is (a) increasing (b) decreasing.

Solution We have

$$f(x) = 4x^3 - 6x^2 - 72x + 30$$

or
$$\begin{aligned} f'(x) &= 12x^2 - 12x - 72 \\ &= 12(x^2 - x - 6) \\ &= 12(x - 3)(x + 2) \end{aligned}$$

Therefore, $f'(x) = 0$ gives $x = -2, 3$. The points $x = -2$ and $x = 3$ divides the real line into three disjoint intervals, namely, $(-\infty, -2)$, $(-2, 3)$ and $(3, \infty)$.

Fig 6.4

In the intervals $(-\infty, -2)$ and $(3, \infty)$, $f'(x)$ is positive while in the interval $(-2, 3)$, $f'(x)$ is negative. Consequently, the function f is increasing in the intervals $(-\infty, -2)$ and $(3, \infty)$ while the function is decreasing in the interval $(-2, 3)$. However, f is neither increasing nor decreasing in $\mathbf{R}$.

Interval	Sign of $f'(x)$	Nature of function f
$(-\infty, -2)$	$(-)\ (-) > 0$	f is increasing
$(-2, 3)$	$(-)\ (+) < 0$	f is decreasing
$(3, \infty)$	$(+)\ (+) > 0$	f is increasing

Example 12 Find intervals in which the function given by $f(x) = \sin 3x$, $x \in \left[0, \frac{\pi}{2}\right]$ is (a) increasing (b) decreasing.

Solution We have

$$f(x) = \sin 3x$$

or

$$f'(x) = 3\cos 3x$$

Therefore, $f'(x) = 0$ gives $\cos 3x = 0$ which in turn gives $3x = \frac{\pi}{2}, \frac{3\pi}{2}$ (as $x \in \left[0, \frac{\pi}{2}\right]$ implies $3x \in \left[0, \frac{3\pi}{2}\right]$). So $x = \frac{\pi}{6}$ and $\frac{\pi}{2}$. The point $x = \frac{\pi}{6}$ divides the interval $\left[0, \frac{\pi}{2}\right]$ into two disjoint intervals $\left[0, \frac{\pi}{6}\right)$ and $\left(\frac{\pi}{6}, \frac{\pi}{2}\right]$.

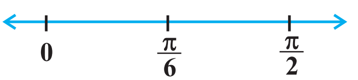


Fig 6.5

Now, $f'(x) > 0$ for all $x \in \left[0, \frac{\pi}{6}\right)$ as $0 \le x < \frac{\pi}{6} \Rightarrow 0 \le 3x < \frac{\pi}{2}$ and $f'(x) < 0$ for all $x \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right)$ as $\frac{\pi}{6} < x < \frac{\pi}{2} \Rightarrow \frac{\pi}{2} < 3x < \frac{3\pi}{2}$.

Therefore, f is increasing in $\left[0, \frac{\pi}{6}\right)$ and decreasing in $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$.

Also, the given function is continuous at $x = 0$ and $x = \frac{\pi}{6}$. Therefore, by Theorem 1, f is increasing on $\left[0, \frac{\pi}{6}\right]$ and decreasing on $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$.

Example 13 Find the intervals in which the function f given by

$$f(x) = \sin x + \cos x,\ 0 \le x \le 2\pi$$

is increasing or decreasing.

Solution We have

$$f(x) = \sin x + \cos x,$$

or $$f'(x) = \cos x - \sin x$$

Now $f'(x) = 0$ gives $\sin x = \cos x$ which gives that $x = \frac{\pi}{4}, \frac{5\pi}{4}$ as $0 \le x \le 2\pi$

The points $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ divide the interval $[0, 2\pi]$ into three disjoint intervals, namely, $\left[0, \frac{\pi}{4}\right), \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, 2\pi\right]$.

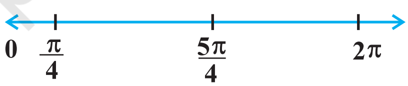


Fig 6.6

Note that $f'(x) > 0$ if $x \in \left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$

or f is increasing in the intervals $\left[0, \frac{\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, 2\pi\right]$

Also $f'(x) < 0$ if $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

or f is decreasing in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

Interval	Sign of $f'(x)$	Nature of function
$\left[0, \frac{\pi}{4}\right)$	> 0	f is increasing
$\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$	< 0	f is decreasing
$\left(\frac{5\pi}{4}, 2\pi\right]$	> 0	f is increasing

EXERCISE 6.2

1. Show that the function given by $f(x) = 3x + 17$ is increasing on **R**.
2. Show that the function given by $f(x) = e^{2x}$ is increasing on **R**.
3. Show that the function given by $f(x) = \sin x$ is

   (a) increasing in $\left(0, \frac{\pi}{2}\right)$ (b) decreasing in $\left(\frac{\pi}{2}, \pi\right)$

   (c) neither increasing nor decreasing in $(0, \pi)$
4. Find the intervals in which the function f given by $f(x) = 2x^2 - 3x$ is

   (a) increasing (b) decreasing
5. Find the intervals in which the function f given by $f(x) = 2x^3 - 3x^2 - 36x + 7$ is

   (a) increasing (b) decreasing
6. Find the intervals in which the following functions are strictly increasing or decreasing:

   (a) $x^2 + 2x - 5$ (b) $10 - 6x - 2x^2$

   (c) $-2x^3 - 9x^2 - 12x + 1$ (d) $6 - 9x - x^2$

   (e) $(x + 1)^3 (x - 3)^3$
7. Show that $y = \log(1+x) - \frac{2x}{2+x}$, $x > -1$, is an increasing function of x throughout its domain.
8. Find the values of x for which $y = [x(x - 2)]^2$ is an increasing function.
9. Prove that $y = \frac{4\sin\theta}{(2+\cos\theta)} - \theta$ is an increasing function of θ in $\left[0, \frac{\pi}{2}\right]$.

10. Prove that the logarithmic function is increasing on $(0, \infty)$.

11. Prove that the function f given by $f(x) = x^2 - x + 1$ is neither strictly increasing nor decreasing on $(-1, 1)$.

12. Which of the following functions are decreasing on $0, \frac{\pi}{2}$?

(A) $\cos x$ (B) $\cos 2x$ (C) $\cos 3x$ (D) $\tan x$

13. On which of the following intervals is the function f given by $f(x) = x^{100} + \sin x - 1$ decreasing ?

(A) (0,1) (B) $\frac{\pi}{2}, \pi$ (C) $0, \frac{\pi}{2}$ (D) None of these

14. For what values of a the function f given by $f(x) = x^2 + ax + 1$ is increasing on [1, 2]?

15. Let I be any interval disjoint from $[-1, 1]$. Prove that the function f given by $f(x) = x + \frac{1}{x}$ is increasing on I.

16. Prove that the function f given by $f(x) = \log \sin x$ is increasing on $\left(0, \frac{\pi}{2}\right)$ and decreasing on $\left(\frac{\pi}{2}, \pi\right)$.

17. Prove that the function f given by $f(x) = \log |\cos x|$ is decreasing on $\left(0, \frac{\pi}{2}\right)$ and increasing on $\left(\frac{3\pi}{2}, 2\pi\right)$.

18. Prove that the function given by $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing in **R**.

19. The interval in which $y = x^2 e^{-x}$ is increasing is

(A) $(-\infty, \infty)$ (B) $(-2, 0)$ (C) $(2, \infty)$ (D) $(0, 2)$

6.4 Maxima and Minima

In this section, we will use the concept of derivatives to calculate the maximum or minimum values of various functions. In fact, we will find the 'turning points' of the graph of a function and thus find points at which the graph reaches its highest (or lowest) *locally*. The knowledge of such points is very useful in sketching the graph of a given function. Further, we will also find the absolute maximum and absolute minimum of a function that are necessary for the solution of many applied problems.

Let us consider the following problems that arise in day to day life.

(i) The profit from a grove of orange trees is given by $P(x) = ax + bx^2$, where a,b are constants and x is the number of orange trees per acre. How many trees per acre will maximise the profit?

(ii) A ball, thrown into the air from a building 60 metres high, travels along a path given by $h(x) = 60 + x - \frac{x^2}{60}$, where x is the horizontal distance from the building and $h(x)$ is the height of the ball . What is the maximum height the ball will reach?

(iii) An Apache helicopter of enemy is flying along the path given by the curve $f(x) = x^2 + 7$. A soldier, placed at the point (1, 2), wants to shoot the helicopter when it is nearest to him. What is the nearest distance?

In each of the above problem, there is something common, i.e., we wish to find out the maximum or minimum values of the given functions. In order to tackle such problems, we first formally define maximum or minimum values of a function, points of local maxima and minima and test for determining such points.

Definition 3 Let f be a function defined on an interval I. Then

(a) f is said to have a *maximum value* in I, if there exists a point c in I such that $f(c) > f(x)$, for all $x \in$ I.

The number $f(c)$ is called the maximum value of f in I and the point c is called a *point of maximum value* of f in I.

(b) f is said to have a minimum value in I, if there exists a point c in I such that $f(c) < f(x)$, for all $x \in$ I.

The number $f(c)$, in this case, is called the minimum value of f in I and the point c, in this case, is called a *point of minimum value* of f in I.

(c) f is said to have an extreme value in I if there exists a point c in I such that $f(c)$ is either a maximum value or a minimum value of f in I.

The number $f(c)$, in this case, is called an *extreme value* of f in I and the point c is called an *extreme point.*

Remark In Fig 6.7(a), (b) and (c), we have exhibited that graphs of certain particular functions help us to find maximum value and minimum value at a point. Infact, through graphs, we can even find maximum/minimum value of a function at a point at which it is not even differentiable (Example 15).

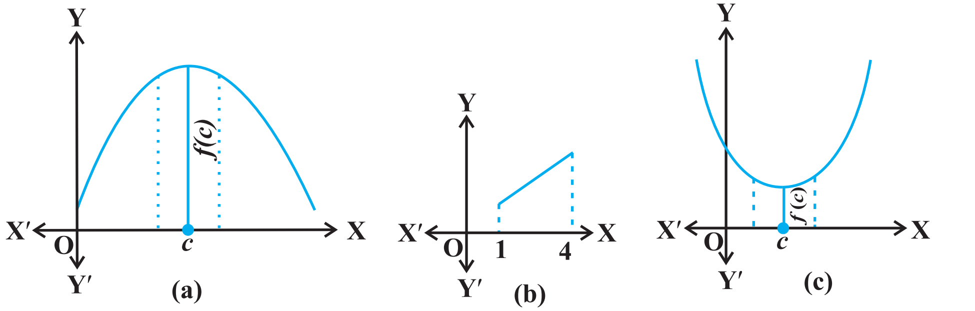


Fig 6.7

Example 14 Find the maximum and the minimum values, if any, of the function f given by

$$f(x) = x^2, x \in \mathbf{R}.$$

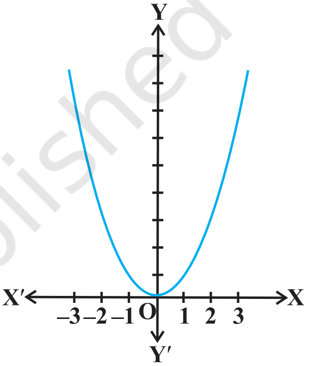


Fig 6.8

Solution From the graph of the given function (Fig 6.8), we have $f(x) = 0$ if $x = 0$. Also

$$f(x) \geq 0, \text{ for all } x \in \mathbf{R}.$$

Therefore, the minimum value of f is 0 and the point of minimum value of f is $x = 0$. Further, it may be observed from the graph of the function that f has no maximum value and hence no point of maximum value of f in **R**.

> **Note** If we restrict the domain of f to [– 2, 1] only, then f will have maximum value$(-2)^2 = 4$ at $x = -2$.

Example 15 Find the maximum and minimum values of f, if any, of the function given by $f(x) = |x|, x \in \mathbf{R}$.

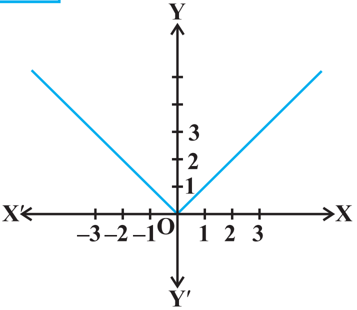


Fig 6.9

Solution From the graph of the given function (Fig 6.9) , note that

$$f(x) \geq 0, \text{ for all } x \in \mathbf{R} \text{ and } f(x) = 0 \text{ if } x = 0.$$

Therefore, the function f has a minimum value 0 and the point of minimum value of f is $x = 0$. Also, the graph clearly shows that f has no maximum value in **R** and hence no point of maximum value in **R**.

> **Note**
>
> (i) If we restrict the domain of f to [– 2, 1] only, then f will have maximum value $|-2| = 2$.

(ii) One may note that the function f in Example 27 is not differentiable at $x = 0$.

Example 16 Find the maximum and the minimum values, if any, of the function given by

$$f(x) = x, x \in (0, 1).$$

Solution The given function is an increasing (strictly) function in the given interval (0, 1). From the graph (Fig 6.10) of the function f, it seems that, it should have the minimum value at a point closest to 0 on its right and the maximum value at a point closest to 1 on its left. Are such points available? Of course, not. It is not possible to locate such points. Infact, if a point x_0 is closest to 0, then we find $\frac{x_0}{2} < x_0$ for all $x_0 \in (0,1)$. Also, if x_1 is closest to 1, then $\frac{x_1 + 1}{2} > x_1$ for all $x_1 \in (0,1)$.

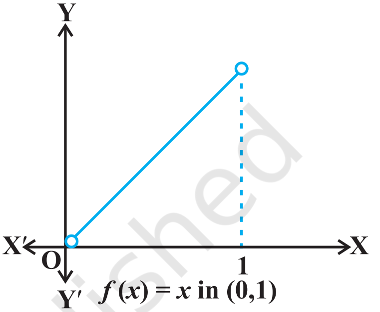


Fig 6.10

Therefore, the given function has neither the maximum value nor the minimum value in the interval (0,1).

Remark The reader may observe that in Example 16, if we include the points 0 and 1 in the domain of f, i.e., if we extend the domain of f to [0,1], then the function f has minimum value 0 at $x = 0$ and maximum value 1 at $x = 1$. Infact, we have the following results (The proof of these results are beyond the scope of the present text)

Every monotonic function assumes its maximum/minimum value at the end points of the domain of definition of the function.

A more general result is

Every continuous function on a closed interval has a maximum and a minimum value.

Note By a monotonic function f in an interval I, we mean that f is either increasing in I or decreasing in I.

Maximum and minimum values of a function defined on a closed interval will be discussed later in this section.

Let us now examine the graph of a function as shown in Fig 6.11. Observe that at points A, B, C and D on the graph, the function changes its nature from decreasing to increasing or vice-versa. These points may be called *turning points* of the given function. Further, observe that at turning points, the graph has either a little hill or a little valley. Roughly speaking, the function has minimum value in some neighbourhood (interval) of each of the points A and C which are at the bottom of their respective

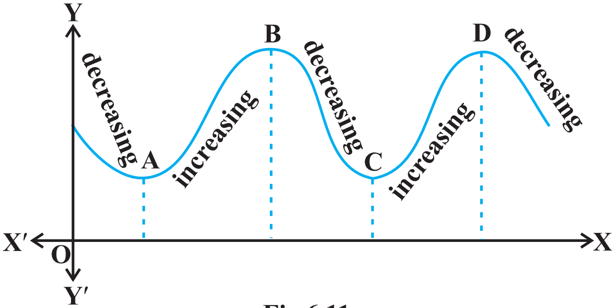


Fig 6.11

valleys. Similarly, the function has maximum value in some neighbourhood of points B and D which are at the top of their respective hills. For this reason, the points A and C may be regarded as points of *local minimum value* (or *relative minimum value*) and points B and D may be regarded as points of *local maximum value* (or *relative maximum value*) for the function. The *local maximum value* and *local minimum value* of the function are referred to as *local maxima* and *local minima*, respectively, of the function.

We now formally give the following definition

Definition 4 Let f be a real valued function and let c be an interior point in the domain of f. Then

(a) c is called a point of *local maxima* if there is an $h > 0$ such that

$$f(c) \geq f(x), \text{ for all } x \text{ in } (c - h, c + h), x \neq c$$

The value $f(c)$ is called the *local maximum value* of f.

(b) c is called a point of *local minima* if there is an $h > 0$ such that

$$f(c) \leq f(x), \text{ for all } x \text{ in } (c - h, c + h)$$

The value $f(\text{c})$ is called the *local minimum value* of f.

Geometrically, the above definition states that if $x = c$ is a point of local maxima of f, then the graph of f around c will be as shown in Fig 6.12(a). Note that the function f is increasing (i.e., $f'(x) > 0$) in the interval $(c - h, c)$ and decreasing (i.e., $f'(x) < 0$) in the interval $(c, c + h)$.

This suggests that $f'(c)$ must be zero.

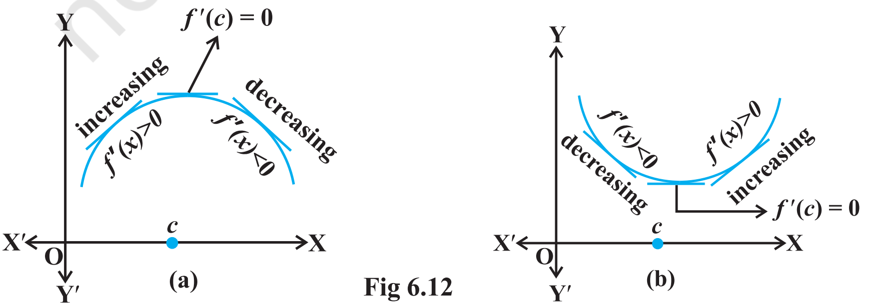


Fig 6.12

Similarly, if c is a point of local minima of f, then the graph of f around c will be as shown in Fig 6.14(b). Here f is decreasing (i.e., $f'(x) < 0$) in the interval $(c - h, c)$ and increasing (i.e., $f'(x) > 0$) in the interval $(c, c + h)$. This again suggest that $f'(c)$ must be zero.

The above discussion lead us to the following theorem (without proof).

Theorem 2 Let f be a function defined on an open interval I. Suppose $c \in$ I be any point. If f has a local maxima or a local minima at $x = c$, then either $f'(c) = 0$ or f is not differentiable at c.

Remark The converse of above theorem need not be true, that is, a point at which the derivative vanishes need not be a point of local maxima or local minima. For example, if $f(x) = x^3$, then $f'(x) = 3x^2$ and so $f'(0) = 0$. But 0 is neither a point of local maxima nor a point of local minima (Fig 6.13).

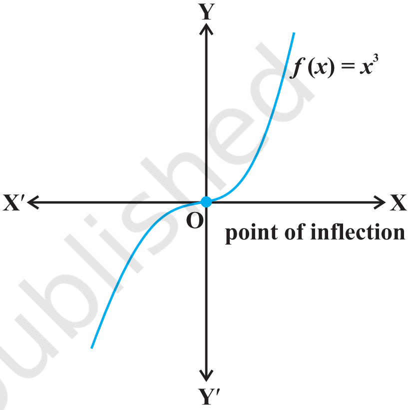


Fig 6.13

Note A point c in the domain of a function f at which either $f'(c) = 0$ or f is not differentiable is called a *critical point* of f. Note that if f is continuous at c and $f'(c) = 0$, then there exists an $h > 0$ such that f is differentiable in the interval $(c - h, c + h)$.

We shall now give a working rule for finding points of local maxima or points of local minima using only the first order derivatives.

Theorem 3 (First Derivative Test) Let f be a function defined on an open interval I. Let f be continuous at a critical point c in I. Then

(i) If $f'(x)$ changes sign from positive to negative as x increases through c, i.e., if $f'(x) > 0$ at every point sufficiently close to and to the left of c, and $f'(x) < 0$ at every point sufficiently close to and to the right of c, then c is a point of *local maxima*.

(ii) If $f'(x)$ changes sign from negative to positive as x increases through c, i.e., if $f'(x) < 0$ at every point sufficiently close to and to the left of c, and $f'(x) > 0$ at every point sufficiently close to and to the right of c, then c is a point of *local minima*.

(iii) If $f'(x)$ does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. Infact, such a point is called *point of inflection* (Fig 6.13).

Note If c is a point of local maxima of f, then $f(c)$ is a local maximum value of f. Similarly, if c is a point of local minima of f, then $f(c)$ is a local minimum value of f.

Figures 6.13 and 6.14, geometrically explain Theorem 3.

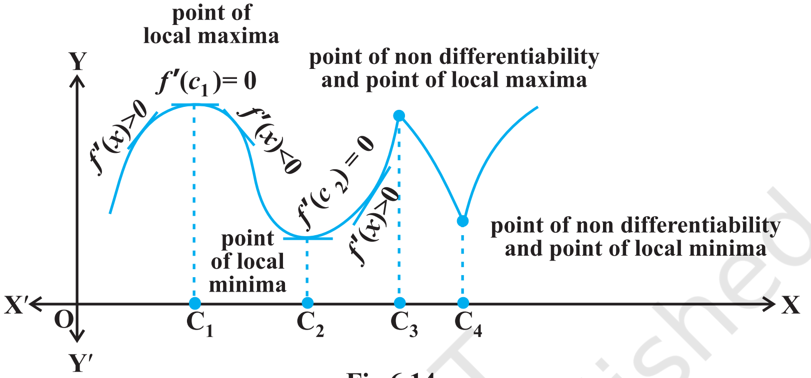


Fig 6.14

Example 17 Find all points of local maxima and local minima of the function f given by

$$f(x) = x^3 - 3x + 3.$$

Solution We have

$$f(x) = x^3 - 3x + 3$$

or $$f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$$

or $$f'(x) = 0 \text{ at } x = 1 \text{ and } x = -1$$

Thus, $x = \pm 1$ are the only critical points which could possibly be the points of local maxima and/or local minima of f. Let us first examine the point $x = 1$.

Note that for values close to 1 and to the right of 1, $f'(x) > 0$ and for values close to 1 and to the left of 1, $f'(x) < 0$. Therefore, by first derivative test, $x = 1$ is a point of local minima and local minimum value is $f(1) = 1$. In the case of $x = -1$, note that $f'(x) > 0$, for values close to and to the left of -1 and $f'(x) < 0$, for values close to and to the right of -1. Therefore, by first derivative test, $x = -1$ is a point of local maxima and local maximum value is $f(-1) = 5$.

Values of x		Sign of $f'(x) = 3(x-1)(x+1)$
Close to 1	to the right (say 1.1 etc.)	>0
	to the left (say 0.9 etc.)	<0
Close to -1	to the right (say -0.9 etc.)	<0
	to the left (say -1.1 etc.)	>0

Example 18 Find all the points of local maxima and local minima of the function f given by

$$f(x) = 2x^3 - 6x^2 + 6x + 5.$$

Solution We have

$$f(x) = 2x^3 - 6x^2 + 6x + 5$$

or $$f'(x) = 6x^2 - 12x + 6 = 6(x-1)^2$$

or $$f'(x) = 0 \quad \text{at} \quad x = 1$$

Thus, $x = 1$ is the only critical point of f. We shall now examine this point for local maxima and/or local minima of f. Observe that $f'(x) \geq 0$, for all $x \in \mathbf{R}$ and in particular $f'(x) > 0$, for values close to 1 and to the left and to the right of 1. Therefore, by first derivative test, the point $x = 1$ is neither a point of local maxima nor a point of local minima. Hence $x = 1$ is a point of inflexion.

Remark One may note that since $f'(x)$, in Example 30, never changes its sign on **R**, graph of f has no turning points and hence no point of local maxima or local minima.

We shall now give another test to examine local maxima and local minima of a given function. This test is often easier to apply than the first derivative test.

Theorem 4 (Second Derivative Test) Let f be a function defined on an interval I and $c \in$ I. Let f be twice differentiable at c. Then

(i) $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$

The value $f(c)$ is local maximum value of f.

(ii) $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$

In this case, $f(c)$ is local minimum value of f.

(iii) The test fails if $f'(c) = 0$ and $f''(c) = 0$.

In this case, we go back to the first derivative test and find whether c is a point of local maxima, local minima or a point of inflexion.

> **Note** As f is twice differentiable at c, we mean second order derivative of f exists at c.

Example 19 Find local minimum value of the function f given by $f(x) = 3 + |x|, x \in \mathbf{R}$.

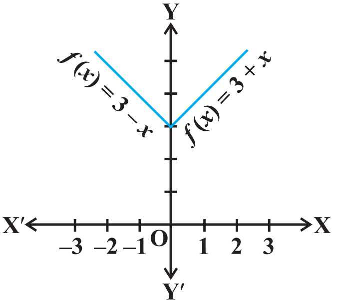


Fig 6.15

Solution Note that the given function is not differentiable at $x = 0$. So, second derivative test fails. Let us try first derivative test. Note that 0 is a critical point of f. Now to the left of 0, $f(x) = 3 - x$ and so $f'(x) = -1 < 0$. Also to the right of 0, $f(x) = 3 + x$ and so $f'(x) = 1 > 0$. Therefore, by first derivative test, $x = 0$ is a point of local minima of f and local minimum value of f is $f(0) = 3$.

Example 20 Find local maximum and local minimum values of the function f given by

$$f(x) = 3x^4 + 4x^3 - 12x^2 + 12$$

Solution We have

$$f(x) = 3x^4 + 4x^3 - 12x^2 + 12$$

or $$f'(x) = 12x^3 + 12x^2 - 24x = 12x(x - 1)(x + 2)$$

or $$f'(x) = 0 \text{ at } x = 0,\ x = 1 \text{ and } x = -2.$$

Now $$f''(x) = 36x^2 + 24x - 24 = 12(3x^2 + 2x - 2)$$

or $$\begin{cases} f''(0) = -24 < 0 \\ f''(1) = 36 > 0 \\ f''(-2) = 72 > 0 \end{cases}$$

Therefore, by second derivative test, $x = 0$ is a point of local maxima and local maximum value of f at $x = 0$ is $f(0) = 12$ while $x = 1$ and $x = -2$ are the points of local minima and local minimum values of f at $x = -1$ and -2 are $f(1) = 7$ and $f(-2) = -20$, respectively.

Example 21 Find all the points of local maxima and local minima of the function f given by

$$f(x) = 2x^3 - 6x^2 + 6x + 5.$$

Solution We have

$$f(x) = 2x^3 - 6x^2 + 6x + 5$$

or $$\begin{cases} f'(x) = 6x^2 - 12x + 6 = 6(x-1)^2 \\ f''(x) = 12(x-1) \end{cases}$$

Now $f'(x) = 0$ gives $x = 1$. Also $f''(1) = 0$. Therefore, the second derivative test fails in this case. So, we shall go back to the first derivative test.

We have already seen (Example 18) that, using first derivative test, $x = 1$ is neither a point of local maxima nor a point of local minima and so it is a point of inflexion.

Example 22 Find two positive numbers whose sum is 15 and the sum of whose squares is minimum.

Solution Let one of the numbers be x. Then the other number is $(15 - x)$. Let $S(x)$ denote the sum of the squares of these numbers. Then

$$S(x) = x^2 + (15 - x)^2 = 2x^2 - 30x + 225$$

or
$$\begin{cases} S'(x) = 4x - 30 \\ S''(x) = 4 \end{cases}$$

Now $S'(x) = 0$ gives $x = \frac{15}{2}$. Also $S''\left(\frac{15}{2}\right) = 4 > 0$. Therefore, by second derivative test, $x = \frac{15}{2}$ is the point of local minima of S. Hence the sum of squares of numbers is minimum when the numbers are $\frac{15}{2}$ and $15 - \frac{15}{2} = \frac{15}{2}$.

Remark Proceeding as in Example 34 one may prove that the two positive numbers, whose sum is k and the sum of whose squares is minimum, are $\frac{k}{2}$ and $\frac{k}{2}$.

Example 23 Find the shortest distance of the point (0, c) from the parabola $y = x^2$, where $\frac{1}{2} \le c \le 5$.

Solution Let (h, k) be any point on the parabola $y = x^2$. Let D be the required distance between (h, k) and $(0, c)$. Then

$$D = \sqrt{(h-0)^2 + (k-c)^2} = \sqrt{h^2 + (k-c)^2} \quad \dots (1)$$

Since (h, k) lies on the parabola $y = x^2$, we have $k = h^2$. So (1) gives

$$D \equiv D(k) = \sqrt{k + (k-c)^2}$$

or
$$D'(k) = \frac{1 + 2(k-c)}{2\sqrt{k + (k-c)^2}}$$

Now
$$D'(k) = 0 \text{ gives } k = \frac{2c-1}{2}$$

Observe that when $k < \frac{2c-1}{2}$, then $2(k-c) + 1 < 0$, i.e., $D'(k) < 0$. Also when $k > \frac{2c-1}{2}$, then $D'(k) > 0$. So, by first derivative test, D (k) is minimum at $k = \frac{2c-1}{2}$.

Hence, the required shortest distance is given by

$$D\left(\frac{2c-1}{2}\right)=\sqrt{\frac{2c-1}{2}+\left(\frac{2c-1}{2}-c\right)^2}=\frac{\sqrt{4c-1}}{2}$$

Note The reader may note that in Example 35, we have used first derivative test instead of the second derivative test as the former is easy and short.

Example 24 Let AP and BQ be two vertical poles at points A and B, respectively. If AP = 16 m, BQ = 22 m and AB = 20 m, then find the distance of a point R on AB from the point A such that $RP^2 + RQ^2$ is minimum.

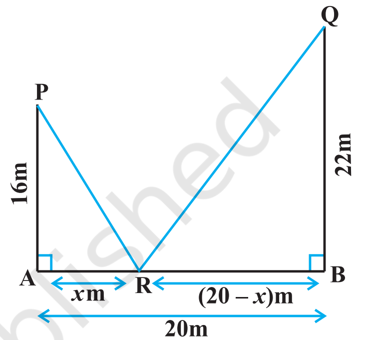


Fig 6.16

Solution Let R be a point on AB such that AR = x m. Then RB = $(20 – x)$ m (as AB = 20 m). From Fig 6.16, we have

$$RP^2 = AR^2 + AP^2$$

and $$RQ^2 = RB^2 + BQ^2$$

Therefore
$$RP^2 + RQ^2 = AR^2 + AP^2 + RB^2 + BQ^2$$
$$= x^2 + (16)^2 + (20 – x)^2 + (22)^2$$
$$= 2x^2 – 40x + 1140$$

Let $$S \equiv S(x) = RP^2 + RQ^2 = 2x^2 – 40x + 1140.$$

Therefore $$S'(x) = 4x – 40.$$

Now $S'(x) = 0$ gives $x = 10$. Also $S''(x) = 4 > 0$, for all x and so $S''(10) > 0$. Therefore, by second derivative test, $x = 10$ is the point of local minima of S. Thus, the distance of R from A on AB is AR = x =10 m.

Example 25 If length of three sides of a trapezium other than base are equal to 10cm, then find the area of the trapezium when it is maximum.

Solution The required trapezium is as given in Fig 6.17. Draw perpendiculars DP and

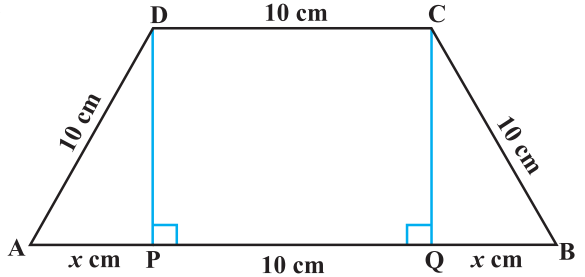


Fig 6.17

CQ on AB. Let AP = x cm. Note that ΔAPD ~ ΔBQC. Therefore, QB = x cm. Also, by Pythagoras theorem, DP = QC = $\sqrt{100-x^2}$. Let A be the area of the trapezium. Then

$$A \equiv A(x) = \frac{1}{2}\text{(sum of parallel sides) (height)}$$

$$= \frac{1}{2}(2x+10+10)\left(\sqrt{100-x^2}\right)$$

$$= (x+10)\left(\sqrt{100-x^2}\right)$$

or
$$A'(x) = (x+10)\frac{(-2x)}{2\sqrt{100-x^2}} + \left(\sqrt{100-x^2}\right)$$

$$= \frac{-2x^2-10x+100}{\sqrt{100-x^2}}$$

Now $\quad A'(x) = 0$ gives $2x^2 + 10x - 100 = 0$, i.e., $x = 5$ and $x = -10$.
Since x represents distance, it can not be negative.
So, $\quad x = 5$. Now

$$A''(x) = \frac{\sqrt{100-x^2}(-4x-10)-(-2x^2-10x+100)\frac{(-2x)}{2\sqrt{100-x^2}}}{100-x^2}$$

$$= \frac{2x^3-300x-1000}{(100-x^2)^{\frac{3}{2}}} \text{ (on simplification)}$$

or
$$A''(5) = \frac{2(5)^3-300(5)-1000}{(100-(5)^2)^{\frac{3}{2}}} = \frac{-2250}{75\sqrt{75}} = \frac{-30}{\sqrt{75}} < 0$$

Thus, area of trapezium is maximum at $x = 5$ and the area is given by

$$A(5) = (5+10)\sqrt{100-(5)^2} = 15\sqrt{75} = 75\sqrt{3}\text{ cm}^2$$

Example 26 Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

Solution Let OC = r be the radius of the cone and OA = h be its height. Let a cylinder with radius OE = x inscribed in the given cone (Fig 6.18). The height QE of the cylinder is given by

$$\frac{\text{QE}}{\text{OA}} = \frac{\text{EC}}{\text{OC}} \quad (\text{since } \Delta\text{QEC} \sim \Delta\text{AOC})$$

or $$\frac{\text{QE}}{h} = \frac{r-x}{r}$$

or $$\text{QE} = \frac{h(r-x)}{r}$$

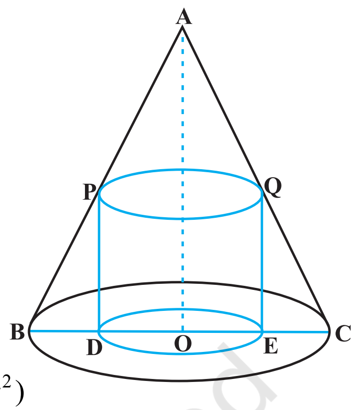


Fig 6.18

Let S be the curved surface area of the given cylinder. Then

$$\text{S} \equiv \text{S}(x) = \frac{2\pi x\,h(r-x)}{r} = \frac{2\pi h}{r}(rx - x^2)$$

or $$\begin{cases} \text{S}'(x) = \dfrac{2\pi h}{r}(r-2x) \\ \text{S}''(x) = \dfrac{-4\pi h}{r} \end{cases}$$

Now $\text{S}'(x) = 0$ gives $x = \frac{r}{2}$. Since $\text{S}''(x) < 0$ for all x, $\text{S}''\left(\frac{r}{2}\right) < 0$. So $x = \frac{r}{2}$ is a point of maxima of S. Hence, the radius of the cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

6.4.1 *Maximum and Minimum Values of a Function in a Closed Interval*

Let us consider a function f given by

$$f(x) = x + 2, x \in (0, 1)$$

Observe that the function is continuous on (0, 1) and neither has a maximum value nor has a minimum value. Further, we may note that the function even has neither a local maximum value nor a local minimum value.

However, if we extend the domain of f to the closed interval [0, 1], then f still may not have a local maximum (minimum) values but it certainly does have maximum value $3 = f(1)$ and minimum value $2 = f(0)$. The maximum value 3 of f at $x = 1$ is called *absolute maximum* value (*global maximum* or *greatest value*) of f on the interval [0, 1]. Similarly, the minimum value 2 of f at $x = 0$ is called the *absolute minimum* value (*global minimum* or *least value*) of f on [0, 1].

Consider the graph given in Fig 6.19 of a continuous function defined on a closed interval $[a, d]$. Observe that the function f has a local minima at $x = b$ and local

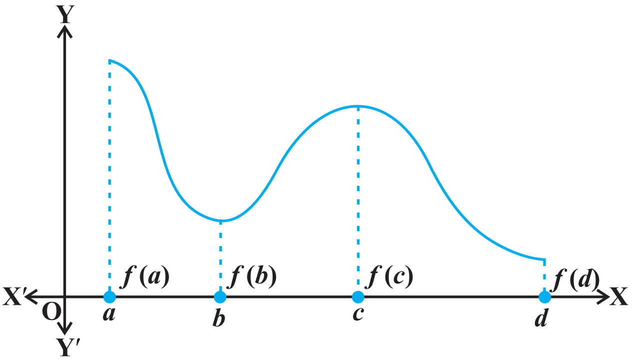


Fig 6.19

minimum value is $f(b)$. The function also has a local maxima at $x = c$ and local maximum value is $f(c)$.

Also from the graph, it is evident that f has absolute maximum value $f(a)$ and absolute minimum value $f(d)$. Further note that the absolute maximum (minimum) value of f is different from local maximum (minimum) value of f.

We will now state two results (without proof) regarding absolute maximum and absolute minimum values of a function on a closed interval I.

Theorem 5 Let f be a continuous function on an interval I = $[a, b]$. Then f has the absolute maximum value and f attains it at least once in I. Also, f has the absolute minimum value and attains it at least once in I.

Theorem 6 Let f be a differentiable function on a closed interval I and let c be any interior point of I. Then

(i) $f'(c) = 0$ if f attains its absolute maximum value at c.

(ii) $f'(c) = 0$ if f attains its absolute minimum value at c.

In view of the above results, we have the following working rule for finding absolute maximum and/or absolute minimum values of a function in a given closed interval $[a, b]$.

Working Rule

Step 1: Find all critical points of f in the interval, i.e., find points x where either $f'(x) = 0$ or f is not differentiable.

Step 2: Take the end points of the interval.

Step 3: At all these points (listed in Step 1 and 2), calculate the values of f.

Step 4: Identify the maximum and minimum values of f out of the values calculated in Step 3. This maximum value will be the absolute maximum (greatest) value of f and the minimum value will be the absolute minimum (least) value of f.

Example 27 Find the absolute maximum and minimum values of a function f given by

$$f(x) = 2x^3 - 15x^2 + 36x + 1 \text{ on the interval } [1, 5].$$

Solution We have

$$f(x) = 2x^3 - 15x^2 + 36x + 1$$

or

$$f'(x) = 6x^2 - 30x + 36 = 6(x-3)(x-2)$$

Note that $f'(x) = 0$ gives $x = 2$ and $x = 3$.

We shall now evaluate the value of f at these points and at the end points of the interval [1, 5], i.e., at $x = 1$, $x = 2$, $x = 3$ and at $x = 5$. So

$$f(1) = 2(1^3) - 15(1^2) + 36(1) + 1 = 24$$
$$f(2) = 2(2^3) - 15(2^2) + 36(2) + 1 = 29$$
$$f(3) = 2(3^3) - 15(3^2) + 36(3) + 1 = 28$$
$$f(5) = 2(5^3) - 15(5^2) + 36(5) + 1 = 56$$

Thus, we conclude that absolute maximum value of f on [1, 5] is 56, occurring at x =5, and absolute minimum value of f on [1, 5] is 24 which occurs at $x = 1$.

Example 28 Find absolute maximum and minimum values of a function f given by

$$f(x) = 12x^{\frac{4}{3}} - 6x^{\frac{1}{3}},\ x \in [-1, 1]$$

Solution We have

$$f(x) = 12x^{\frac{4}{3}} - 6x^{\frac{1}{3}}$$

or

$$f'(x) = 16x^{\frac{1}{3}} - \frac{2}{x^{\frac{2}{3}}} = \frac{2(8x-1)}{x^{\frac{2}{3}}}$$

Thus, $f'(x) = 0$ gives $x = \frac{1}{8}$. Further note that $f'(x)$ is not defined at $x = 0$. So the critical points are $x = 0$ and $x = \frac{1}{8}$. Now evaluating the value of f at critical points $x = 0, \frac{1}{8}$ and at end points of the interval $x = -1$ and $x = 1$, we have

$$f(-1) = 12(-1)^{\frac{4}{3}} - 6(-1)^{\frac{1}{3}} = 18$$
$$f(0) = 12(0) - 6(0) = 0$$

$$f\left(\frac{1}{8}\right) = 12\left(\frac{1}{8}\right)^{\frac{4}{3}} - 6\left(\frac{1}{8}\right)^{\frac{1}{3}} = \frac{-9}{4}$$

$$f(1) = 12(1)^{\frac{4}{3}} - 6(1)^{\frac{1}{3}} = 6$$

Hence, we conclude that absolute maximum value of f is 18 that occurs at $x = -1$ and absolute minimum value of f is $\frac{-9}{4}$ that occurs at $x = \frac{1}{8}$.

Example 29 An Apache helicopter of enemy is flying along the curve given by $y = x^2 + 7$. A soldier, placed at (3, 7), wants to shoot down the helicopter when it is nearest to him. Find the nearest distance.

Solution For each value of x, the helicopter's position is at point $(x, x^2 + 7)$. Therefore, the distance between the helicopter and the soldier placed at (3,7) is

$$\sqrt{(x-3)^2 + (x^2+7-7)^2}, \text{ i.e., } \sqrt{(x-3)^2 + x^4}.$$

Let $\qquad f(x) = (x - 3)^2 + x^4$

or $\qquad f'(x) = 2(x - 3) + 4x^3 = 2(x - 1)(2x^2 + 2x + 3)$

Thus, $f'(x) = 0$ gives $x = 1$ or $2x^2 + 2x + 3 = 0$ for which there are no real roots. Also, there are no end points of the interval to be added to the set for which f' is zero, i.e., there is only one point, namely, $x = 1$. The value of f at this point is given by $f(1) = (1 - 3)^2 + (1)^4 = 5$. Thus, the distance between the solider and the helicopter is $\sqrt{f(1)} = \sqrt{5}$.

Note that $\sqrt{5}$ is either a maximum value or a minimum value. Since

$$\sqrt{f(0)} = \sqrt{(0-3)^2 + (0)^4} = 3 > \sqrt{5},$$

it follows that $\sqrt{5}$ is the minimum value of $\sqrt{f(x)}$. Hence, $\sqrt{5}$ is the minimum distance between the soldier and the helicopter.

EXERCISE 6.3

1. Find the maximum and minimum values, if any, of the following functions given by
   (i) $f(x) = (2x - 1)^2 + 3$ (ii) $f(x) = 9x^2 + 12x + 2$
   (iii) $f(x) = -(x - 1)^2 + 10$ (iv) $g(x) = x^3 + 1$

2. Find the maximum and minimum values, if any, of the following functions given by

(i) $f(x) = |x + 2| - 1$ (ii) $g(x) = -|x + 1| + 3$

(iii) $h(x) = \sin(2x) + 5$ (iv) $f(x) = |\sin 4x + 3|$

(v) $h(x) = x + 1, x \in (-1, 1)$

3. Find the local maxima and local minima, if any, of the following functions. Find also the local maximum and the local minimum values, as the case may be:

(i) $f(x) = x^2$ (ii) $g(x) = x^3 - 3x$

(iii) $h(x) = \sin x + \cos x, 0 < x < \dfrac{\pi}{2}$

(iv) $f(x) = \sin x - \cos x,\ 0 < x < 2\pi$

(v) $f(x) = x^3 - 6x^2 + 9x + 15$ (vi) $g(x) = \dfrac{x}{2} + \dfrac{2}{x},\ x > 0$

(vii) $g(x) = \dfrac{1}{x^2 + 2}$ (viii) $f(x) = x\sqrt{1 - x},\ 0 < x < 1$

4. Prove that the following functions do not have maxima or minima:

(i) $f(x) = e^x$ (ii) $g(x) = \log x$

(iii) $h(x) = x^3 + x^2 + x + 1$

5. Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals:

(i) $f(x) = x^3, x \in [-2, 2]$ (ii) $f(x) = \sin x + \cos x,\ x \in [0, \pi]$

(iii) $f(x) = 4x - \dfrac{1}{2}x^2,\ x \in \left[-2, \dfrac{9}{2}\right]$ (iv) $f(x) = (x - 1)^2 + 3,\ x \in [-3, 1]$

6. Find the maximum profit that a company can make, if the profit function is given by

$$p(x) = 41 - 72x - 18x^2$$

7. Find both the maximum value and the minimum value of

$3x^4 - 8x^3 + 12x^2 - 48x + 25$ on the interval [0, 3].

8. At what points in the interval $[0, 2\pi]$, does the function $\sin 2x$ attain its maximum value?

9. What is the maximum value of the function $\sin x + \cos x$?

10. Find the maximum value of $2x^3 - 24x + 107$ in the interval [1, 3]. Find the maximum value of the same function in $[-3, -1]$.

11. It is given that at $x = 1$, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value, on the interval [0, 2]. Find the value of a.

12. Find the maximum and minimum values of $x + \sin 2x$ on $[0, 2\pi]$.

13. Find two numbers whose sum is 24 and whose product is as large as possible.

14. Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximum.

15. Find two positive numbers x and y such that their sum is 35 and the product $x^2 y^5$ is a maximum.

16. Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.

17. A square piece of tin of side 18 cm is to be made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible.

18. A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum ?

19. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.

20. Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.

21. Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimetres, find the dimensions of the can which has the minimum surface area?

22. A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?

23. Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

24. Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ time the radius of the base.

25. Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

26. Show that semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$.

Choose the correct answer in Questions 27 and 29.

27. The point on the curve $x^2 = 2y$ which is nearest to the point (0, 5) is

(A) $(2\sqrt{2},4)$ (B) $(2\sqrt{2},0)$ (C) $(0, 0)$ (D) $(2, 2)$

28. For all real values of x, the minimum value of $\frac{1-x+x^2}{1+x+x^2}$ is

(A) 0 (B) 1 (C) 3 (D) $\frac{1}{3}$

29. The maximum value of $[x(x-1)+1]^{\frac{1}{3}}$, $0 \le x \le 1$ is

(A) $\left(\frac{1}{3}\right)^{\frac{1}{3}}$ (B) $\frac{1}{2}$ (C) 1 (D) 0

Miscellaneous Examples

Example 30 A car starts from a point P at time $t = 0$ seconds and stops at point Q. The distance x, in metres, covered by it, in t seconds is given by

$$x = t^2\left(2 - \frac{t}{3}\right)$$

Find the time taken by it to reach Q and also find distance between P and Q.

Solution Let v be the velocity of the car at t seconds.

Now $$x = t^2\left(2 - \frac{t}{3}\right)$$

Therefore $$v = \frac{dx}{dt} = 4t - t^2 = t(4 - t)$$

Thus, $v = 0$ gives $t = 0$ and/or $t = 4$.

Now $v = 0$ at P as well as at Q and at P, $t = 0$. So, at Q, $t = 4$. Thus, the car will reach the point Q after 4 seconds. Also the distance travelled in 4 seconds is given by

$$x]_{t=4} = 4^2\left(2 - \frac{4}{3}\right) = 16\left(\frac{2}{3}\right) = \frac{32}{3}\text{ m}$$

Example 31 A water tank has the shape of an inverted right circular cone with its axis vertical and vertex lowermost. Its semi-vertical angle is $\tan^{-1}(0.5)$. Water is poured into it at a constant rate of 5 cubic metre per hour. Find the rate at which the level of the water is rising at the instant when the depth of water in the tank is 4 m.

Solution Let r, h and α be as in Fig 6.20. Then $\tan\alpha = \frac{r}{h}$.

So $\alpha = \tan^{-1}\left(\frac{r}{h}\right)$.

But $\alpha = \tan^{-1}(0.5)$ (given)

or $\frac{r}{h} = 0.5$

or $r = \frac{h}{2}$

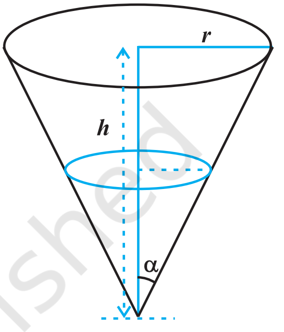


Fig 6.20

Let V be the volume of the cone. Then

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12}$$

Therefore $\frac{dV}{dt} = \frac{d}{dh}\left(\frac{\pi h^3}{12}\right)\cdot\frac{dh}{dt}$ (by Chain Rule)

$$= \frac{\pi}{4}h^2\frac{dh}{dt}$$

Now rate of change of volume, i.e., $\frac{dV}{dt} = 5\ \text{m}^3/\text{h}$ and $h = 4$ m.

Therefore $5 = \frac{\pi}{4}(4)^2\cdot\frac{dh}{dt}$

or $\frac{dh}{dt} = \frac{5}{4\pi} = \frac{35}{88}$ m/h $\left(\pi = \frac{22}{7}\right)$

Thus, the rate of change of water level is $\frac{35}{88}$ m/h.

Example 32 A man of height 2 metres walks at a uniform speed of 5 km/h away from a lamp post which is 6 metres high. Find the rate at which the length of his shadow increases.

Solution In Fig 6.21, Let AB be the lamp-post, the lamp being at the position B and let MN be the man at a particular time t and let AM = l metres. Then, MS is the shadow of the man. Let MS = s metres.

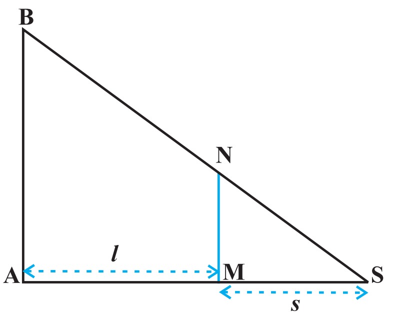


Fig 6.21

Note that $\Delta MSN \sim \Delta ASB$

or $$\frac{MS}{AS} = \frac{MN}{AB}$$

or AS = 3s (as MN = 2 and AB = 6 (given))

Thus AM = 3s – s = 2s. But AM = l

So l = 2s

Therefore $$\frac{dl}{dt} = 2\frac{ds}{dt}$$

Since $\frac{dl}{dt} = 5$ km/h. Hence, the length of the shadow increases at the rate $\frac{5}{2}$ km/h.

Example 33 Find intervals in which the function given by

$$f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$$

is (a) increasing (b) decreasing.

Solution We have

$$f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$$

Therefore $$f'(x) = \frac{3}{10}(4x^3) - \frac{4}{5}(3x^2) - 3(2x) + \frac{36}{5}$$

$$= \frac{6}{5}(x-1)(x+2)(x-3) \quad \text{(on simplification)}$$

Now $f'(x) = 0$ gives $x = 1$, $x = -2$, or $x = 3$. The points $x = 1, -2$, and 3 divide the real line into four disjoint intervals namely, $(-\infty, -2)$, $(-2, 1)$, $(1, 3)$ and $(3, \infty)$ (Fig 6.22).

Fig 6.22

Consider the interval $(-\infty, -2)$, i.e., when $-\infty < x < -2$.

In this case, we have $x - 1 < 0$, $x + 2 < 0$ and $x - 3 < 0$.

(In particular, observe that for $x = -3$, $f'(x) = (x - 1)(x + 2)(x - 3) = (-4)(-1)(-6) < 0$)

Therefore, $\quad f'(x) < 0$ when $-\infty < x < -2$.

Thus, the function f is decreasing in $(-\infty, -2)$.

Consider the interval $(-2, 1)$, i.e., when $-2 < x < 1$.

In this case, we have $x - 1 < 0$, $x + 2 > 0$ and $x - 3 < 0$

(In particular, observe that for $x = 0$, $f'(x) = (x - 1)(x + 2)(x - 3) = (-1)(2)(-3) = 6 > 0$)

So $\quad f'(x) > 0$ when $-2 < x < 1$.

Thus, $\quad f$ is increasing in $(-2, 1)$.

Now consider the interval $(1, 3)$, i.e., when $1 < x < 3$. In this case, we have $x - 1 > 0$, $x + 2 > 0$ and $x - 3 < 0$.

So, $\quad f'(x) < 0$ when $1 < x < 3$.

Thus, $\quad f$ is decreasing in $(1, 3)$.

Finally, consider the interval $(3, \infty)$, i.e., when $x > 3$. In this case, we have $x - 1 > 0$, $x + 2 > 0$ and $x - 3 > 0$. So $f'(x) > 0$ when $x > 3$.

Thus, f is increasing in the interval $(3, \infty)$.

Example 34 Show that the function f given by

$$f(x) = \tan^{-1}(\sin x + \cos x),\ x > 0$$

is always an increasing function in $\left(0, \frac{\pi}{4}\right)$.

Solution We have

$$f(x) = \tan^{-1}(\sin x + \cos x),\ x > 0$$

Therefore

$$f'(x) = \frac{1}{1 + (\sin x + \cos x)^2}(\cos x - \sin x)$$

$$= \frac{\cos x - \sin x}{2 + \sin 2x} \qquad \text{(on simplification)}$$

Note that 2 + sin 2x > 0 for all x in $\left(0, \frac{\pi}{4}\right)$.

Therefore $\quad f'(x) > 0$ if $\cos x - \sin x > 0$

or $\quad f'(x) > 0$ if $\cos x > \sin x$ or $\cot x > 1$

Now $\quad \cot x > 1$ if $\tan x < 1$, i.e., if $0 < x < \frac{\pi}{4}$

Thus $\quad f'(x) > 0$ in $\left(0, \frac{\pi}{4}\right)$

Hence f is increasing function in $\left(0, \frac{\pi}{4}\right)$.

Example 35 A circular disc of radius 3 cm is being heated. Due to expansion, its radius increases at the rate of 0.05 cm/s. Find the rate at which its area is increasing when radius is 3.2 cm.

Solution Let r be the radius of the given disc and A be its area. Then

$$A = \pi r^2$$

or
$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} \qquad \text{(by Chain Rule)}$$

Now approximate rate of increase of radius = $dr = \frac{dr}{dt}\Delta t = 0.05$ cm/s.

Therefore, the approximate rate of increase in area is given by

$$dA = \frac{dA}{dt}(\Delta t) = 2\pi r\left(\frac{dr}{dt}\Delta t\right)$$

$$= 2\pi\,(3.2)\,(0.05) = 0.320\pi \text{ cm}^2/\text{s} \quad (r = 3.2 \text{ cm})$$

Example 36 An open topped box is to be constructed by removing equal squares from each corner of a 3 metre by 8 metre rectangular sheet of aluminium and folding up the sides. Find the volume of the largest such box.

Solution Let x metre be the length of a side of the removed squares. Then, the height of the box is x, length is $8 - 2x$ and breadth is $3 - 2x$ (Fig 6.23). If $V(x)$ is the volume of the box, then

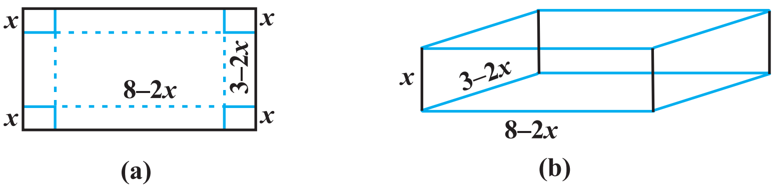


Fig 6.23

$$V(x) = x(3 - 2x)(8 - 2x)$$
$$= 4x^3 - 22x^2 + 24x$$

Therefore
$$\begin{cases} V'(x) = 12x^2 - 44x + 24 = 4(x-3)(3x-2) \\ V''(x) = 24x - 44 \end{cases}$$

Now $V'(x) = 0$ gives $x = 3, \frac{2}{3}$. But $x \neq 3$ (Why?)

Thus, we have $x = \frac{2}{3}$. Now $V''\left(\frac{2}{3}\right) = 24\left(\frac{2}{3}\right) - 44 = -28 < 0$.

Therefore, $x = \frac{2}{3}$ is the point of maxima, i.e., if we remove a square of side $\frac{2}{3}$ metre from each corner of the sheet and make a box from the remaining sheet, then the volume of the box such obtained will be the largest and it is given by

$$V\left(\frac{2}{3}\right) = 4\left(\frac{2}{3}\right)^3 - 22\left(\frac{2}{3}\right)^2 + 24\left(\frac{2}{3}\right)$$
$$= \frac{200}{27} \text{ m}^3$$

Example 37 Manufacturer can sell x items at a price of rupees $\left(5 - \frac{x}{100}\right)$ each. The cost price of x items is Rs $\left(\frac{x}{5} + 500\right)$. Find the number of items he should sell to earn maximum profit.

Solution Let S (x) be the selling price of x items and let C (x) be the cost price of x items. Then, we have

$$S(x) = \left(5 - \frac{x}{100}\right)x = 5x - \frac{x^2}{100}$$

and
$$C(x) = \frac{x}{5} + 500$$

Thus, the profit function P (x) is given by

$$P(x) = S(x) - C(x) = 5x - \frac{x^2}{100} - \frac{x}{5} - 500$$

i.e.
$$P(x) = \frac{24}{5}x - \frac{x^2}{100} - 500$$

or
$$P'(x) = \frac{24}{5} - \frac{x}{50}$$

Now $P'(x) = 0$ gives $x = 240$. Also $P''(x) = \frac{-1}{50}$. So $P''(240) = \frac{-1}{50} < 0$

Thus, $x = 240$ is a point of maxima. Hence, the manufacturer can earn maximum profit, if he sells 240 items.

Miscellaneous Exercise on Chapter 6

1. Show that the function given by $f(x) = \frac{\log x}{x}$ has maximum at $x = e$.

2. The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base ?

3. Find the intervals in which the function f given by

$$f(x) = \frac{4\sin x - 2x - x\cos x}{2 + \cos x}$$

is (i) increasing (ii) decreasing.

4. Find the intervals in which the function f given by $f(x) = x^3 + \frac{1}{x^3}$, $x \neq 0$ is

(i) increasing (ii) decreasing.

5. Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with its vertex at one end of the major axis.

6. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3. If building of tank costs Rs 70 per sq metres for the base and Rs 45 per square metre for sides. What is the cost of least expensive tank?

7. The sum of the perimeter of a circle and square is k, where k is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.

8. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

9. A point on the hypotenuse of a triangle is at distance a and b from the sides of the triangle.

Show that the minimum length of the hypotenuse is $\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}}$.

10. Find the points at which the function f given by $f(x) = (x-2)^4 (x+1)^3$ has

(i) local maxima (ii) local minima

(iii) point of inflexion

11. Find the absolute maximum and minimum values of the function f given by

$$f(x) = \cos^2 x + \sin x,\ x \in [0, \pi]$$

12. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.

13. Let f be a function defined on $[a, b]$ such that $f'(x) > 0$, for all $x \in (a, b)$. Then prove that f is an increasing function on (a, b).

14. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

15. Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and semi vertical angle α is one-third that of the cone and the greatest volume of cylinder is $\frac{4}{27}\pi h^3 \tan^2 \alpha$.

16. A cylindrical tank of radius 10 m is being filled with wheat at the rate of 314 cubic metre per hour. Then the depth of the wheat is increasing at the rate of

(A) 1 m/h (B) 0.1 m/h

(C) 1.1 m/h (D) 0.5 m/h

Summary

- If a quantity y varies with another quantity x, satisfying some rule $y = f(x)$, then $\frac{dy}{dx}$ (or $f'(x)$) represents the rate of change of y with respect to x and $\left.\frac{dy}{dx}\right]_{x=x_0}$ (or $f'(x_0)$) represents the rate of change of y with respect to x at $x = x_0$.
- If two variables x and y are varying with respect to another variable t, i.e., if $x = f(t)$ and $y = g(t)$, then by Chain Rule

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}, \text{ if } \frac{dx}{dt} \neq 0.$$

- A function f is said to be
  (a) increasing on an interval (a, b) if
  $x_1 < x_2$ in $(a, b) \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in (a, b)$.
  Alternatively, if $f'(x) \geq 0$ for each x in (a, b)
  (b) decreasing on (a,b) if
  $x_1 < x_2$ in $(a, b) \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in (a, b)$.
  (c) constant in (a, b), if $f(x) = c$ for all $x \in (a, b)$, where c is a constant.
- A point c in the domain of a function f at which either $f'(c) = 0$ or f is not differentiable is called a *critical point* of f.
- ***First Derivative Test*** Let f be a function defined on an open interval I. Let f be continuous at a critical point c in I. Then
  (i) If $f'(x)$ changes sign from positive to negative as x increases through c, i.e., if $f'(x) > 0$ at every point sufficiently close to and to the left of c, and $f'(x) < 0$ at every point sufficiently close to and to the right of c, then c is a point of *local maxima*.

(ii) If $f'(x)$ changes sign from negative to positive as x increases through c, i.e., if $f'(x) < 0$ at every point sufficiently close to and to the left of c, and $f'(x) > 0$ at every point sufficiently close to and to the right of c, then c is a point of *local minima*.

(iii) If $f'(x)$ does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. Infact, such a point is called *point of inflexion*.

◆ ***Second Derivative Test*** Let f be a function defined on an interval I and $c \in$ I. Let f be twice differentiable at c. Then

(i) $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$

The values $f(c)$ is local maximum value of f.

(ii) $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$

In this case, $f(c)$ is local minimum value of f.

(iii) The test fails if $f'(c) = 0$ and $f''(c) = 0$.

In this case, we go back to the first derivative test and find whether c is a point of maxima, minima or a point of inflexion.

◆ Working rule for finding absolute maxima and/or absolute minima

Step 1: Find all critical points of f in the interval, i.e., find points x where either $f'(x) = 0$ or f is not differentiable.

Step 2: Take the end points of the interval.

Step 3: At all these points (listed in Step 1 and 2), calculate the values of f.

Step 4: Identify the maximum and minimum values of f out of the values calculated in Step 3. This maximum value will be the absolute maximum value of f and the minimum value will be the absolute minimum value of f.

Appendix 1

PROOFS IN MATHEMATICS

❖ *Proofs are to Mathematics what calligraphy is to poetry.*
Mathematical works do consist of proofs just as
poems do consist of characters.
— VLADIMIR ARNOLD ❖

A.1.1 Introduction

In Classes IX, X and XI, we have learnt about the concepts of a statement, compound statement, negation, converse and contrapositive of a statement; axioms, conjectures, theorems and deductive reasoning.

Here, we will discuss various methods of proving mathematical propositions.

A.1.2 What is a Proof?

Proof of a mathematical statement consists of sequence of statements, each statement being justified with a definition or an axiom or a proposition that is previously established by the method of deduction using only the allowed logical rules.

Thus, each proof is a chain of deductive arguments each of which has its premises and conclusions. Many a times, we prove a proposition directly from what is given in the proposition. But some times it is easier to prove an equivalent proposition rather than proving the proposition itself. This leads to, two ways of proving a proposition directly or indirectly and the proofs obtained are called direct proof and indirect proof and further each has three different ways of proving which is discussed below.

Direct Proof It is the proof of a proposition in which we directly start the proof with what is given in the proposition.

(i) **Straight forward approach** It is a chain of arguments which leads directly from what is given or assumed, with the help of axioms, definitions or already proved theorems, to what is to be proved using rules of logic.

Consider the following example:

Example 1 Show that if $x^2 - 5x + 6 = 0$, then $x = 3$ or $x = 2$.

Solution $x^2 - 5x + 6 = 0$ (given)

$\Rightarrow$ $(x-3)(x-2)=0$ (replacing an expression by an equal/equivalent expression)

$\Rightarrow$ $x-3=0$ or $x-2=0$ (from the established theorem $ab=0 \Rightarrow$ either $a=0$ or $b=0$, for a, b in **R**)

$\Rightarrow$ $x-3+3=0+3$ or $x-2+2=0+2$ (adding equal quantities on either side of the equation does not alter the nature of the equation)

$\Rightarrow$ $x+0=3$ or $x+0=2$ (using the identity property of integers under addition)

$\Rightarrow$ $x=3$ or $x=2$ (using the identity property of integers under addition)

Hence, $x^2-5x+6=0$ implies $x=3$ or $x=2$.

Explanation Let p be the given statement "$x^2-5x+6=0$" and q be the conclusion statement "$x=3$ or $x=2$".

From the statement p, we deduced the statement r: "$(x-3)(x-2)=0$" by replacing the expression x^2-5x+6 in the statement p by another expression $(x-3)(x-2)$ which is equal to x^2-5x+6.

There arise two questions:

(i) How does the expression $(x-3)(x-2)$ is equal to the expression x^2-5x+6?

(ii) How can we replace an expression with another expression which is equal to the former?

The first one is proved in earlier classes by factorization, i.e.,

$x^2-5x+6=x^2-3x-2x+6=x(x-3)-2(x-3)=(x-3)(x-2)$.

The second one is by valid form of argumentation (rules of logic)

Next this statement r becomes premises or given and deduce the statement s "$x-3=0$ or $x-2=0$" and the reasons are given in the brackets.

This process continues till we reach the conclusion.

The symbolic equivalent of the argument is to prove by deduction that $p \Rightarrow q$ is true.

Starting with p, we deduce $p \Rightarrow r \Rightarrow s \Rightarrow \ldots \Rightarrow q$. This implies that "$p \Rightarrow q$" is true.

Example 2 Prove that the function $f: \mathbf{R} \rightarrow \mathbf{R}$ defined by $f(x)=2x+5$ is one-one.

Solution Note that a function f is one-one if

$$f(x_1)=f(x_2) \Rightarrow x_1=x_2 \quad \text{(definition of one-one function)}$$

Now, given that $f(x_1)=f(x_2)$, i.e., $2x_1+5=2x_2+5$

$\Rightarrow$ $2x_1+5-5=2x_2+5-5$ (adding the same quantity on both sides)

$$\Rightarrow \qquad 2x_1 + 0 = 2x_2 + 0$$

$$\Rightarrow \qquad 2x_1 = 2x_2 \text{ (using additive identity of real number)}$$

$$\Rightarrow \qquad \frac{2}{2}x_1 = \frac{2}{2}x_2 \text{ (dividing by the same non zero quantity)}$$

$$\Rightarrow \qquad x_1 = x_2$$

Hence, the given function is one-one.

(ii) Mathematical Induction

Mathematical induction, is a strategy, of proving a proposition which is deductive in nature. The whole basis of proof of this method depends on the following axiom:

For a given subset S of **N**, if

(i) the natural number $1 \in S$ and

(ii) the natural number $k + 1 \in S$ whenever $k \in S$, then $S = \mathbf{N}$.

According to the principle of mathematical induction, if a statement "S(n) is true for $n = 1$" (or for some starting point j), and if "S(n) is true for $n = k$" implies that "S(n) is true for $n = k + 1$" (whatever integer $k \geq j$ may be), then the statement is true for any positive integer n, for all $n \geq j$.

We now consider some examples.

Example 3 Show that if

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \text{ then } A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$$

Solution We have

$$P(n) : A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$$

We note that

$$P(1) : A^1 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Therefore, P(1) is true.

Assume that P(k) is true, i.e.,

$$P(k) : A^k = \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$$

We want to prove that P(k + 1) is true whenever P(k) is true, i.e.,

$$P(k+1) : A^{k+1} = \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$$

Now $$A^{k+1} = A^k \,.\, A$$

Since P(k) is true, we have

$$A^{k+1} = \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos k\theta\cos\theta - \sin k\theta\sin\theta & \cos k\theta\sin\theta + \sin k\theta\cos\theta \\ -\sin k\theta\cos\theta - \cos k\theta\sin\theta & -\sin k\theta\sin\theta + \cos k\theta\cos\theta \end{bmatrix}$$

(by matrix multiplication)

$$= \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, P(n) is true for all $n \geq 1$ (by the principle of mathematical induction).

(iii) Proof by cases or by exhaustion

This method of proving a statement $p \Rightarrow q$ is possible only when p can be split into several cases, r, s, t (say) so that $p = r \vee s \vee t$ (where "$\vee$" is the symbol for "OR").

If the conditionals $r \Rightarrow q$;

$s \Rightarrow q$;

and $t \Rightarrow q$

are proved, then $(r \vee s \vee t) \Rightarrow q$, is proved and so $p \Rightarrow q$ is proved.

The method consists of examining every possible case of the hypothesis. It is practically convenient only when the number of possible cases are few.

Example 4 Show that in any triangle ABC,

$$a = b\cos C + c\cos B$$

Solution Let p be the statement "ABC is any triangle" and q be the statement "$a = b\cos C + c\cos B$"

Let ABC be a triangle. From A draw AD a perpendicular to BC (BC produced if necessary).

As we know that any triangle has to be either acute or obtuse or right angled, we can split p into three statements r, s and t, where *r* : ABC is an acute angled triangle with ∠ C is acute.

s : ABC is an obtuse angled triangle with ∠ C is obtuse.

t : ABC is a right angled triangle with ∠ C is right angle.

Hence, we prove the theorem by three cases.

Case (i) When ∠ C is acute (Fig. A1.1).

From the right angled triangle ADB,

$$\frac{BD}{AB} = \cos B$$

i.e. $BD = AB \cos B$

$= c \cos B$

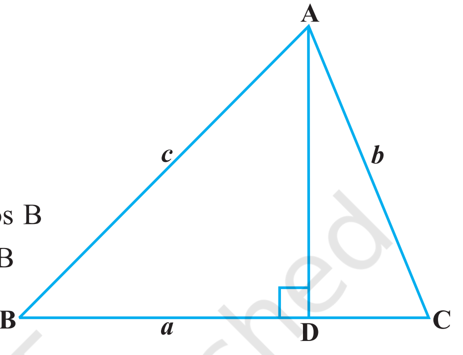


Fig A1.1

From the right angled triangle ADC,

$$\frac{CD}{AC} = \cos C$$

i.e. $CD = AC \cos C$

$= b \cos C$

Now $a = BD + CD$

$= c \cos B + b \cos C$ ... (1)

Case (ii) When ∠ C is obtuse (Fig A1.2).

From the right angled triangle ADB,

$$\frac{BD}{AB} = \cos B$$

i.e. $BD = AB \cos B$

$= c \cos B$

From the right angled triangle ADC,

$$\frac{CD}{AC} = \cos \angle ACD$$

$= \cos (180° – C)$

$= – \cos C$

i.e. $CD = – AC \cos C$

$= – b \cos C$

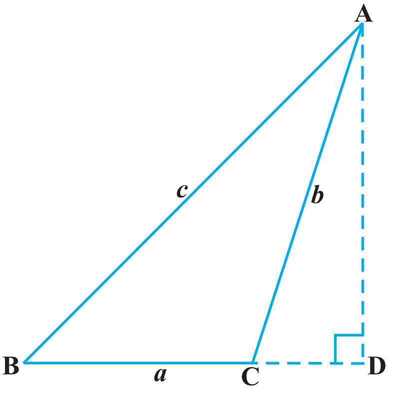


Fig A1.2

Now $a = \text{BC} = \text{BD} - \text{CD}$

i.e. $a = c \cos \text{B} - (-b \cos \text{C})$

$$a = c \cos \text{B} + b \cos \text{C} \qquad \ldots (2)$$

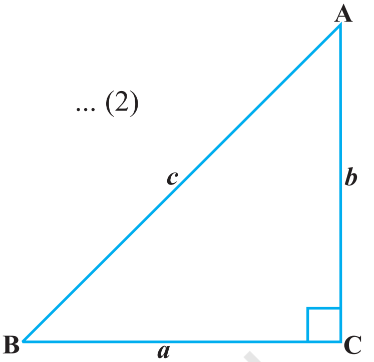


Fig A1.3

Case (iii) When $\angle$ C is a right angle (Fig A1.3).

From the right angled triangle ACB,

$$\frac{\text{BC}}{\text{AB}} = \cos \text{B}$$

i.e. $\text{BC} = \text{AB} \cos \text{B}$

$$a = c \cos \text{B},$$

and $b \cos \text{C} = b \cos 90^0 = 0.$

Thus, we may write $a = 0 + c \cos \text{B}$

$$= b \cos \text{C} + c \cos \text{B} \qquad \ldots (3)$$

From (1), (2) and (3). We assert that for any triangle ABC,

$$a = b \cos \text{C} + c \cos \text{B}$$

By case (i), $r \Rightarrow q$ is proved.

By case (ii), $s \Rightarrow q$ is proved.

By case (iii), $t \Rightarrow q$ is proved.

Hence, from the proof by cases, $(r \vee s \vee t) \Rightarrow q$ is proved, i.e., $p \Rightarrow q$ is proved.

Indirect Proof Instead of proving the given proposition directly, we establish the proof of the proposition through proving a proposition which is equivalent to the given proposition.

(i) Proof by contradiction (*Reductio Ad Absurdum*) : Here, we start with the assumption that the given statement is false. By rules of logic, we arrive at a conclusion contradicting the assumption and hence it is inferred that the assumption is wrong and hence the given statement is true.

Let us illustrate this method by an example.

Example 5 Show that the set of all prime numbers is infinite.

Solution Let P be the set of all prime numbers. We take the negation of the statement "the set of all prime numbers is infinite", i.e., we assume the set of all prime numbers to be finite. Hence, we can list all the prime numbers as $\text{P}_1, \text{P}_2, \text{P}_3, \ldots, \text{P}_k$ (say). Note that we have assumed that there is no prime number other than $\text{P}_1, \text{P}_2, \text{P}_3, \ldots, \text{P}_k$.

Now consider $\text{N} = (\text{P}_1 \, \text{P}_2 \, \text{P}_3 \ldots \text{P}_k) + 1 \ldots (1)$

N is not in the list as N is larger than any of the numbers in the list.

N is either prime or composite.

If N is a prime, then by (1), there exists a prime number which is not listed.

On the other hand, if N is composite, it should have a prime divisor. But none of the numbers in the list can divide N, because they all leave the remainder 1. Hence, the prime divisor should be other than the one in the list.

Thus, in both the cases whether N is a prime or a composite, we ended up with contradiction to the fact that we have listed all the prime numbers.

Hence, our assumption that set of all prime numbers is finite is false.

Thus, the set of all prime numbers is infinite.

Note Observe that the above proof also uses the method of proof by cases.

(ii) Proof by using contrapositive statement of the given statement

Instead of proving the conditional $p \Rightarrow q$, we prove its equivalent, i.e., $\sim q \Rightarrow \sim p$. (students can verify).

The contrapositive of a conditional can be formed by interchanging the conclusion and the hypothesis and negating both.

Example 6 Prove that the function $f : \mathbf{R} \rightarrow \mathbf{R}$ defined by $f(x) = 2x + 5$ is one-one.

Solution A function is one-one if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Using this we have to show that "$2x_1 + 5 = 2x_2 + 5$" $\Rightarrow$ "$x_1 = x_2$". This is of the form $p \Rightarrow q$, where, p is $2x_1 + 5 = 2x_2 + 5$ and $q : x_1 = x_2$. We have proved this in Example 2 of "direct method".

We can also prove the same by using contrapositive of the statement. Now contrapositive of this statement is $\sim q \Rightarrow \sim p$, i.e., contrapositive of " if $f(x_1) = f(x_2)$, then $x_1 = x_2$" is "if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$".

$$
\begin{aligned}
\text{Now} \quad & x_1 \neq x_2 \\
\Rightarrow \quad & 2x_1 \neq 2x_2 \\
\Rightarrow \quad & 2x_1 + 5 \neq 2x_2 + 5 \\
\Rightarrow \quad & f(x_1) \neq f(x_2).
\end{aligned}
$$

Since "$\sim q \Rightarrow \sim p$", is equivalent to "$p \Rightarrow q$" the proof is complete.

Example 7 Show that "if a matrix A is invertible, then A is non singular".

Solution Writing the above statement in symbolic form, we have
$p \Rightarrow q$, where, p is "matrix A is invertible" and q is "A is non singular"

Instead of proving the given statement, we prove its contrapositive statement, i.e., if A is not a non singular matrix, then the matrix A is not invertible.

If A is not a non singular matrix, then it means the matrix A is singular, i.e.,

$$|A| = 0$$

Then $$A^{-1} = \frac{adj\ A}{|A|} \text{ does not exist as } |A| = 0$$

Hence, A is not invertible.

Thus, we have proved that if A is not a non singular matrix, then A is not invertible. i.e., $\sim q \Rightarrow \sim p$.

Hence, if a matrix A is invertible, then A is non singular.

(iii) Proof by a counter example

In the history of Mathematics, there are occasions when all attempts to find a valid proof of a statement fail and the uncertainty of the truth value of the statement remains unresolved.

In such a situation, it is beneficial, if we find an example to falsify the statement. The example to disprove the statement is called a *counter example*. Since the disproof of a proposition $p \Rightarrow q$ is merely a proof of the proposition $\sim (p \Rightarrow q)$. Hence, this is also a method of proof.

Example 8 For each $n, 2^{2^n} + 1$ is a prime ($n \in \mathbf{N}$).

This was once thought to be true on the basis that

$$2^{2^1} + 1 = 2^2 + 1 = 5 \text{ is a prime.}$$

$$2^{2^2} + 1 = 2^4 + 1 = 17 \text{ is a prime.}$$

$$2^{2^3} + 1 = 2^8 + 1 = 257 \text{ is a prime.}$$

However, at first sight the generalisation looks to be correct. But, eventually it was shown that $$2^{2^5} + 1 = 2^{32} + 1 = 4294967297$$

which is not a prime since $4294967297 = 641 \times 6700417$ (a product of two numbers).

So the generalisation "For each n, $2^{2^n} + 1$ is a prime ($n \in \mathbf{N}$)" is false.

Just this one example $2^{2^5} + 1$ is sufficient to disprove the generalisation. This is the counter example.

Thus, we have proved that the generalisation "For each $n, 2^{2^n} + 1$ is a prime ($n \in \mathbf{N}$)" is not true in general.

Example 9 Every continuous function is differentiable.

Proof We consider some functions given by

(i) $f(x) = x^2$

(ii) $g(x) = e^x$

(iii) $h(x) = \sin x$

These functions are continuous for all values of x. If we check for their differentiability, we find that they are all differentiable for all the values of x. This makes us to believe that the generalisation "Every continuous function is differentiable" may be true. But if we check the differentiability of the function given by "$\phi(x) = |x|$" which is continuous, we find that it is not differentiable at $x = 0$. This means that the statement "Every continuous function is differentiable" is false, in general. Just this one function "$\phi(x) = |x|$" is sufficient to disprove the statement. Hence, "$\phi(x) = |x|$" is called a counter example to disprove "Every continuous function is differentiable".

Appendix 2

MATHEMATICAL MODELLING

A.2.1 Introduction

In class XI, we have learnt about mathematical modelling as an attempt to study some part (or form) of some real-life problems in mathematical terms, i.e., the conversion of a physical situation into mathematics using some suitable conditions. Roughly speaking mathematical modelling is an activity in which we make models to describe the behaviour of various phenomenal activities of our interest in many ways using words, drawings or sketches, computer programs, mathematical formulae etc.

In earlier classes, we have observed that solutions to many problems, involving applications of various mathematical concepts, involve mathematical modelling in one way or the other. Therefore, it is important to study mathematical modelling as a separate topic.

In this chapter, we shall further study mathematical modelling of some real-life problems using techniques/results from matrix, calculus and linear programming.

A.2.2 Why Mathematical Modelling?

Students are aware of the solution of word problems in arithmetic, algebra, trigonometry and linear programming etc. Sometimes we solve the problems without going into the physical insight of the situational problems. Situational problems need physical insight that is **introduction** of physical laws and some symbols to compare the mathematical results obtained with practical values. To solve many problems faced by us, we need a technique and this is what is known as *mathematical modelling*. Let us consider the following problems:

(i) To find the width of a river (particularly, when it is difficult to cross the river).

(ii) To find the optimal angle in case of shot-put (by considering the variables such as : the height of the thrower, resistance of the media, acceleration due to gravity etc.).

(iii) To find the height of a tower (particularly, when it is not possible to reach the top of the tower).

(iv) To find the temperature at the surface of the Sun.

(v) Why heart patients are not allowed to use lift? (without knowing the physiology of a human being).

(vi) To find the mass of the Earth.

(vii) Estimate the yield of pulses in India from the standing crops (a person is not allowed to cut all of it).

(viii) Find the volume of blood inside the body of a person (a person is not allowed to bleed completely).

(ix) Estimate the population of India in the year 2020 (a person is not allowed to wait till then).

All of these problems can be solved and infact have been solved with the help of Mathematics using mathematical modelling. In fact, you might have studied the methods for solving some of them in the present textbook itself. However, it will be instructive if you first try to solve them yourself and that too without the help of Mathematics, if possible, you will then appreciate the power of Mathematics and the need for mathematical modelling.

A.2.3 Principles of Mathematical Modelling

Mathematical modelling is a principled activity and so it has some principles behind it. These principles are almost philosophical in nature. Some of the basic principles of mathematical modelling are listed below in terms of instructions:

(i) Identify the need for the model. (for what we are looking for)

(ii) List the parameters/variables which are required for the model.

(iii) Identify the available relevent data. (what is given?)

(iv) Identify the circumstances that can be applied (assumptions)

(v) Identify the governing physical principles.

(vi) Identify

 (a) the equations that will be used.

 (b) the calculations that will be made.

 (c) the solution which will follow.

(vii) Identify tests that can check the

 (a) consistency of the model.

 (b) utility of the model.

(viii) Identify the parameter values that can improve the model.

The above principles of mathematical modelling lead to the following: steps for mathematical modelling.

Step 1: Identify the physical situation.

Step 2: Convert the physical situation into a mathematical model by introducing parameters / variables and using various known physical laws and symbols.

Step 3: Find the solution of the mathematical problem.

Step 4: Interpret the result in terms of the original problem and compare the result with observations or experiments.

Step 5: If the result is in good agreement, then accept the model. Otherwise modify the hypotheses / assumptions according to the physical situation and go to Step 2.

The above steps can also be viewed through the following diagram:

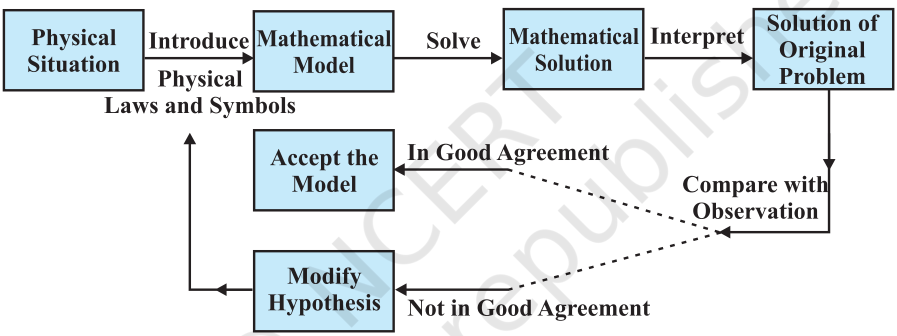


Fig A.2.1

Example 1 Find the height of a given tower using mathematical modelling.

Solution Step 1 Given physical situation is "to find the height of a given tower".

Step 2 Let AB be the given tower (Fig A.2.2). Let PQ be an observer measuring the height of the tower with his eye at P. Let $PQ = h$ and let height of tower be H. Let α be the angle of elevation from the eye of the observer to the top of the tower.

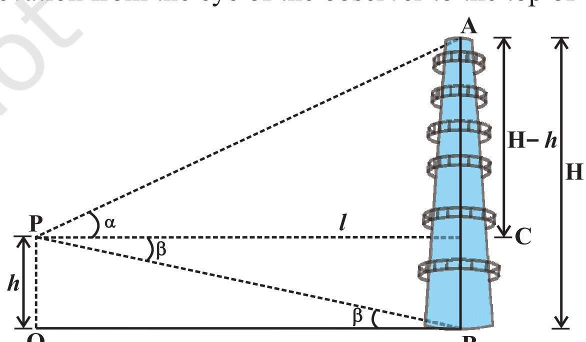


Fig A.2.2

Let $$l = PC = QB$$

Now $$\tan\alpha = \frac{AC}{PC} = \frac{H-h}{l}$$

or $$H = h + l\tan\alpha \qquad \text{... (1)}$$

Step 3 Note that the values of the parameters h, l and α (using sextant) are known to the observer and so (1) gives the solution of the problem.

Step 4 In case, if the foot of the tower is not accessible, i.e., when l is not known to the observer, let β be the angle of depression from P to the foot B of the tower. So from ΔPQB, we have

$$\tan\beta = \frac{PQ}{QB} = \frac{h}{l} \text{ or } l = h\cot\beta$$

Step 5 is not required in this situation as exact values of the parameters h, l, α and β are known.

Example 2 Let a business firm produces three types of products P_1, P_2 and P_3 that uses three types of raw materials R_1, R_2 and R_3. Let the firm has purchase orders from two clients F_1 and F_2. Considering the situation that the firm has a limited quantity of R_1, R_2 and R_3, respectively, prepare a model to determine the quantities of the raw material R_1, R_2 and R_3 required to meet the purchase orders.

Solution Step 1 The physical situation is well identified in the problem.

Step 2 Let A be a matrix that represents purchase orders from the two clients F_1 and F_2. Then, A is of the form

$$A = \begin{array}{c} \\ F_1 \\ F_2 \end{array}\begin{array}{c} \begin{array}{ccc} P_1 & P_2 & P_3 \end{array} \\ \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} \end{array}$$

Let B be the matrix that represents the amount of raw materials R_1, R_2 and R_3, required to manufacture each unit of the products P_1, P_2 and P_3. Then, B is of the form

$$B = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \end{array}\begin{array}{c} \begin{array}{ccc} R_1 & R_2 & R_3 \end{array} \\ \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} \end{array}$$

Step 3 Note that the product (which in this case is well defined) of matrices A and B is given by the following matrix

$$AB = \begin{matrix} & R_1 & R_2 & R_3 \\ F_1 & \bullet & \bullet & \bullet \\ F_2 & \bullet & \bullet & \bullet \end{matrix}$$

which in fact gives the desired quantities of the raw materials R_1, R_2 and R_3 to fulfill the purchase orders of the two clients F_1 and F_2.

Example 3 Interpret the model in Example 2, in case

$$A = \begin{bmatrix} 10 & 15 & 6 \\ 10 & 20 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 3 & 4 & 0 \\ 7 & 9 & 3 \\ 5 & 12 & 7 \end{bmatrix}$$

and the available raw materials are 330 units of R_1, 455 units of R_2 and 140 units of R_3.

Solution Note that

$$AB = \begin{bmatrix} 10 & 15 & 6 \\ 10 & 20 & 0 \end{bmatrix} \begin{bmatrix} 3 & 4 & 0 \\ 7 & 9 & 3 \\ 5 & 12 & 7 \end{bmatrix}$$

$$= \begin{matrix} & R_1 & R_2 & R_3 \\ F_1 & 165 & 247 & 87 \\ F_2 & 170 & 220 & 60 \end{matrix}$$

This clearly shows that to meet the purchase order of F_1 and F_2, the raw material required is 335 units of R_1, 467 units of R_2 and 147 units of R_3 which is much more than the available raw material. Since the amount of raw material required to manufacture each unit of the three products is fixed, we can either ask for an increase in the available raw material or we may ask the clients to reduce their orders.

Remark If we replace A in Example 3 by A_1 given by

$$A_1 = \begin{bmatrix} 9 & 12 & 6 \\ 10 & 20 & 0 \end{bmatrix}$$

i.e., if the clients agree to reduce their purchase orders, then

$$A_1 B = \begin{bmatrix} 9 & 12 & 6 \\ 10 & 20 & 0 \end{bmatrix} \begin{bmatrix} 3 & 4 & 0 \\ 7 & 9 & 3 \\ 5 & 12 & 7 \end{bmatrix} = \begin{bmatrix} 141 & 216 & 78 \\ 170 & 220 & 60 \end{bmatrix}$$

This requires 311 units of R_1, 436 units of R_2 and 138 units of R_3 which are well below the available raw materials, i.e., 330 units of R_1, 455 units of R_2 and 140 units of R_3. Thus, if the revised purchase orders of the clients are given by A_1, then the firm can easily supply the purchase orders of the two clients.

Note One may further modify A so as to make full use of the available raw material.

Query Can we make a mathematical model with a given B and with fixed quantities of the available raw material that can help the firm owner to ask the clients to modify their orders in such a way that the firm makes the full use of its available raw material?

The answer to this query is given in the following example:

Example 4 Suppose P_1, P_2, P_3 and R_1, R_2, R_3 are as in Example 2. Let the firm has 330 units of R_1, 455 units of R_2 and 140 units of R_3 available with it and let the amount of raw materials R_1, R_2 and R_3 required to manufacture each unit of the three products is given by

$$B = \begin{matrix} & \begin{matrix} R_1 & R_2 & R_3 \end{matrix} \\ \begin{matrix} P_1 \\ P_2 \\ P_3 \end{matrix} & \begin{bmatrix} 3 & 4 & 0 \\ 7 & 9 & 3 \\ 5 & 12 & 7 \end{bmatrix} \end{matrix}$$

How many units of each product is to be made so as to utilise the full available raw material?

Solution Step 1 The situation is easily identifiable.

Step 2 Suppose the firm produces x units of P_1, y units of P_2 and z units of P_3. Since product P_1 requires 3 units of R_1, P_2 requires 7 units of R_1 and P_3 requires 5 units of R_1 (observe matrix B) and the total number of units, of R_1, available is 330, we have

$$3x + 7y + 5z = 330 \text{ (for raw material } R_1)$$

Similarly, we have

$$4x + 9y + 12z = 455 \text{ (for raw material } R_2)$$

and

$$3y + 7z = 140 \text{ (for raw material } R_3)$$

This system of equations can be expressed in matrix form as

$$\begin{bmatrix} 3 & 7 & 5 \\ 4 & 9 & 12 \\ 0 & 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 330 \\ 455 \\ 140 \end{bmatrix}$$

Step 3 Using elementary row operations, we obtain

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ 35 \\ 5 \end{bmatrix}$$

This gives $x = 20$, $y = 35$ and $z = 5$. Thus, the firm can produce 20 units of P_1, 35 units of P_2 and 5 units of P_3 to make full use of its available raw material.

Remark One may observe that if the manufacturer decides to manufacture according to the available raw material and not according to the purchase orders of the two clients F_1 and F_2 (as in Example 3), he/she is unable to meet these purchase orders as F_1 demanded 6 units of P_3 where as the manufacturer can make only 5 units of P_3.

Example 5 A manufacturer of medicines is preparing a production plan of medicines M_1 and M_2. There are sufficient raw materials available to make 20000 bottles of M_1 and 40000 bottles of M_2, but there are only 45000 bottles into which either of the medicines can be put. Further, it takes 3 hours to prepare enough material to fill 1000 bottles of M_1, it takes 1 hour to prepare enough material to fill 1000 bottles of M_2 and there are 66 hours available for this operation. The profit is Rs 8 per bottle for M_1 and Rs 7 per bottle for M_2. How should the manufacturer schedule his/her production in order to maximise profit?

Solution Step 1 To find the number of bottles of M_1 and M_2 in order to maximise the profit under the given hypotheses.

Step 2 Let x be the number of bottles of type M_1 medicine and y be the number of bottles of type M_2 medicine. Since profit is Rs 8 per bottle for M_1 and Rs 7 per bottle for M_2, therefore the objective function (which is to be maximised) is given by

$$Z \equiv Z(x, y) = 8x + 7y$$

The objective function is to be maximised subject to the constraints (Refer Chapter 12 on Linear Programming)

$$\left.\begin{array}{l} x \le 20000 \\ y \le 40000 \\ x + y \le 45000 \\ 3x + y \le 66000 \\ x \ge 0, y \ge 0 \end{array}\right\} \quad \ldots (1)$$

Step 3 The shaded region OPQRST is the feasible region for the constraints (1) (Fig A.2.3). The co-ordinates of vertices O, P, Q, R, S and T are (0, 0), (20000, 0), (20000, 6000), (10500, 34500), (5000, 40000) and (0, 40000), respectively.

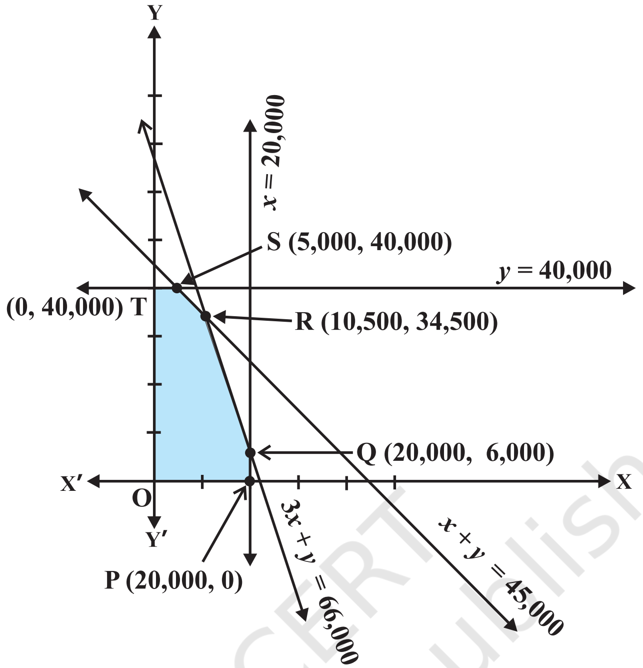


Fig A.2.3

Note that

Z at P (0, 0) = 0

Z at P (20000, 0) = 8 × 20000 = 160000

Z at Q (20000, 6000) = 8 × 20000 + 7 × 6000 = 202000

Z at R (10500, 34500) = 8 × 10500 + 7 × 34500 = 325500

Z at S = (5000, 40000) = 8 × 5000 + 7 × 40000 = 320000

Z at T = (0, 40000) = 7 × 40000 = 280000

Now observe that the profit is maximum at x = 10500 and y = 34500 and the maximum profit is ₹325500. Hence, the manufacturer should produce 10500 bottles of M_1 medicine and 34500 bottles of M_2 medicine in order to get maximum profit of ₹325500.

Example 6 Suppose a company plans to produce a new product that incur some costs (fixed and variable) and let the company plans to sell the product at a fixed price. Prepare a mathematical model to examine the profitability.

Solution Step 1 Situation is clearly identifiable.

Step 2 Formulation: We are given that the costs are of two types: fixed and variable. The fixed costs are independent of the number of units produced (e.g., rent and rates), while the variable costs increase with the number of units produced (e.g., material). Initially, we assume that the variable costs are directly proportional to the number of units produced — this should simplify our model. The company earn a certain amount of money by selling its products and wants to ensure that it is maximum. For convenience, we assume that all units produced are sold immediately.

The mathematical model

Let x = number of units produced and sold

C = total cost of production (in rupees)

I = income from sales (in rupees)

P = profit (in rupees)

Our assumptions above state that C consists of two parts:

(i) fixed cost = a (in rupees),

(ii) variable cost = b (rupees/unit produced).

Then $$C = a + bx \qquad \text{... (1)}$$

Also, income I depends on selling price s (rupees/unit)

Thus $$I = sx \qquad \text{... (2)}$$

The profit P is then the difference between income and costs. So

$$\begin{aligned} P &= I - C \\ &= sx - (a + bx) \\ &= (s - b)\,x - a \qquad \text{... (3)} \end{aligned}$$

We now have a mathematical model of the relationships (1) to (3) *between the variables x,* C, I, P*, a, b, s.* These variables may be classified as:

independent	x
dependent	C, I, P
parameters	a, b, s

The manufacturer, knowing x, a, b, s can determine P.

Step 3 From (3), we can observe that for the break even point (i.e., make neither profit nor loss), he must have P = 0, i.e., $x = \dfrac{a}{s-b}$ units.

Steps 4 and 5 In view of the break even point, one may conclude that if the company produces few units, i.e., less than $x = \dfrac{a}{s-b}$ units, then the company will suffer loss and if it produces large number of units, i.e., much more than $\frac{a}{s-b}$ units, then it can make huge profit. Further, if the break even point proves to be unrealistic, then another model could be tried or the assumptions regarding cash flow may be modified.

Remark From (3), we also have

$$\frac{dP}{dx} = s - b$$

This means that rate of change of P with respect to x depends on the quantity $s - b$, which is the difference of selling price and the variable cost of each product. Thus, in order to gain profit, this should be positive and to get large gains, we need to produce large quantity of the product and at the same time try to reduce the variable cost.

Example 7 Let a tank contains 1000 litres of brine which contains 250 g of salt per litre. Brine containing 200 g of salt per litre flows into the tank at the rate of 25 litres per minute and the mixture flows out at the same rate. Assume that the mixture is kept uniform all the time by stirring. What would be the amount of salt in the tank at any time t?

Solution Step 1 The situation is easily identifiable.

Step 2 Let $y = y(t)$ denote the amount of salt (in kg) in the tank at time t (in minutes) after the inflow, outflow starts. Further assume that y is a differentiable function.

When $t = 0$, i.e., before the inflow–outflow of the brine starts,

$$y = 250 \text{ g} \times 1000 = 250 \text{ kg}$$

Note that the change in y occurs due to the inflow, outflow of the mixture.

Now the inflow of brine brings salt into the tank at the rate of 5 kg per minute (as 25×200 g = 5 kg) and the outflow of brine takes salt out of the tank at the rate of

$25\left(\frac{y}{1000}\right) = \frac{y}{40}$ kg per minute (as at time t, the salt in the tank is $\frac{y}{1000}$ kg).

Thus, the rate of change of salt with respect to t is given by

$$\frac{dy}{dt} = 5 - \frac{y}{40} \qquad \text{(Why?)}$$

or

$$\frac{dy}{dt} + \frac{1}{40}y = 5 \qquad \text{... (1)}$$

This gives a mathematical model for the given problem.

Step 3 Equation (1) is a linear equation and can be easily solved. The solution of (1) is given by

$$y e^{\frac{t}{40}} = 200 e^{\frac{t}{40}} + \mathrm{C} \text{ or } y\,(t) = 200 + \mathrm{C}\, e^{-\frac{t}{40}} \qquad \dots (2)$$

where, c is the constant of integration.

Note that when $t = 0$, $y = 250$. Therefore, $250 = 200 + \mathrm{C}$

or $\mathrm{C} = 50$

Then (2) reduces to

$$y = 200 + 50\, e^{-\frac{t}{40}} \qquad \dots (3)$$

or $$\frac{y-200}{50} = e^{-\frac{t}{40}}$$

or $$e^{\frac{t}{40}} = \frac{50}{y-200}$$

Therefore $$t = 40 \log_e \left(\frac{50}{y-200}\right) \qquad \dots (4)$$

Here, the equation (4) gives the time t at which the salt in tank is y kg.

Step 4 Since $e^{-\frac{t}{40}}$ is always positive, from (3), we conclude that $y > 200$ at all times Thus, the minimum amount of salt content in the tank is 200 kg.

Also, from (4), we conclude that $t > 0$ if and only if $0 < y - 200 < 50$ i.e., if and only if $200 < y < 250$ i.e., the amount of salt content in the tank after the start of inflow and outflow of the brine is between 200 kg and 250 kg.

Limitations of Mathematical Modelling

Till today many mathematical models have been developed and applied successfully to understand and get an insight into thousands of situations. Some of the subjects like mathematical physics, mathematical economics, operations research, bio-mathematics etc. are almost synonymous with mathematical modelling.

But there are still a large number of situations which are yet to be modelled. The reason behind this is that either the situation are found to be very complex or the mathematical models formed are mathematically intractable.

The development of the powerful computers and super computers has enabled us to mathematically model a large number of situations (even complex situations). Due to these fast and advanced computers, it has been possible to prepare more realistic models which can obtain better agreements with observations.

However, we do not have good guidelines for choosing various parameters / variables and also for estimating the values of these parameters / variables used in a mathematical model. Infact, we can prepare reasonably accurate models to fit any data by choosing five or six parameters / variables. We require a minimal number of parameters / variables to be able to estimate them accurately.

Mathematical modelling of large or complex situations has its own special problems. These type of situations usually occur in the study of world models of environment, oceanography, pollution control etc. Mathematical modellers from all disciplines — mathematics, computer science, physics, engineering, social sciences, etc., are involved in meeting these challenges with courage.