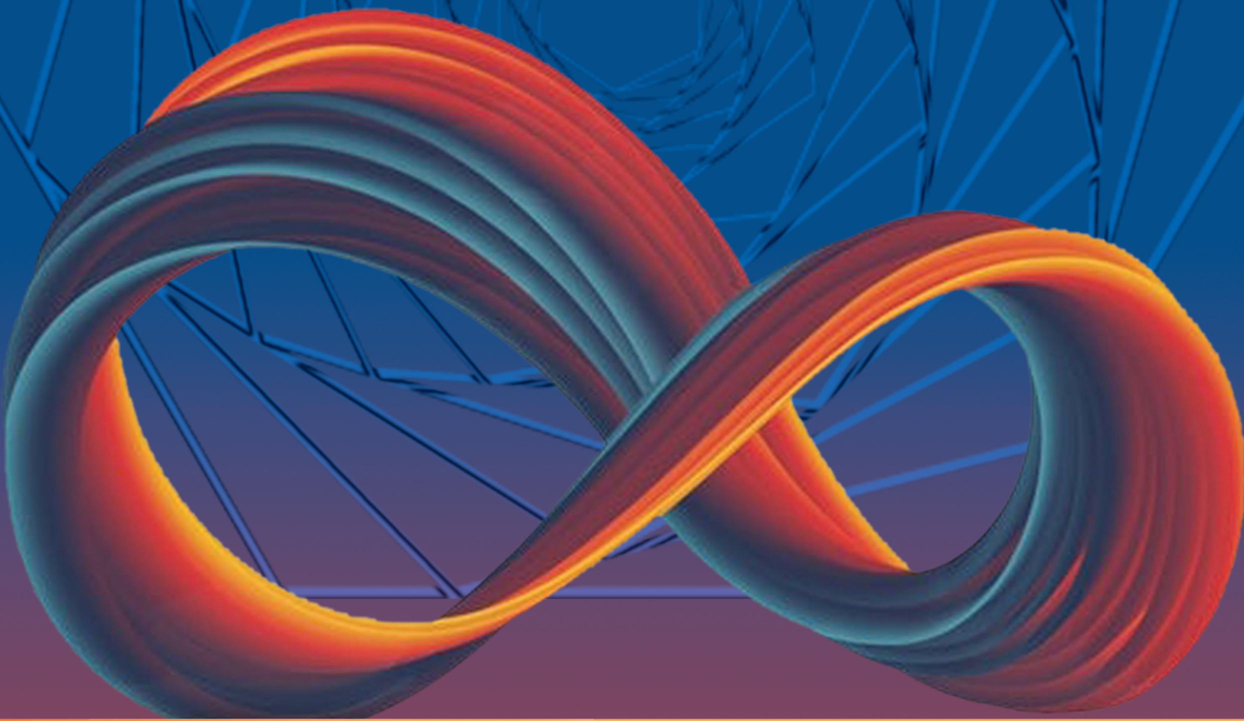
Telangana State Board of
INTERMEDIATE Education
FIRST YEAR

MATHEMATICS - IA



BASIC
LEARNING MATERIAL

For The Academic Year : 2021-2022

TELANGANA STATE BOARD OF INTERMEDIATE EDUCATION

MATHEMATICS - IA

FIRST YEAR

(English Medium)

BASIC LEARNING MATERIAL

ACADEMIC YEAR
2021-2022

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PREFACE

The ongoing Global Pandemic Covid-19 that has engulfed the entire world has changed every sphere of our life. Education, of course is not an exception. In the absence of Physical Classroom Teaching, Department of Intermediate Education Telangana has successfully engaged the students and imparted education through TV lessons. In the back drop of the unprecedented situation due to the pandemic TSBIE has reduced the burden of curriculum load by considering only 70% syllabus for class room instruction as well as for the forthcoming Intermediate Examinations. It has also increased the choice of questions in the examination pattern for the convenience of the students.

To cope up with exam fear and stress and to prepare the students for annual exams in such a short span of time , TSBIE has prepared “Basic Learning Material” that serves as a primer for the students to face the examinations confidently. It must be noted here that, the Learning Material is not comprehensive and can never substitute the Textbook. At most it gives guidance as to how the students should include the essential steps in their answers and build upon them. I wish you to utilize the Basic Learning Material after you have thoroughly gone through the Text Book so that it may enable you to reinforce the concepts that you have learnt from the Textbook and Teachers. I appreciate ERTW Team, Subject Experts, who have involved day in and out to come out with the Basic Learning Material in such a short span of time.

I would appreciate the feedback from all the stake holders for enriching the learning material and making it cent percent error free in all aspects.

The material can also be accessed through our website**www.tsbie.cgg.gov.in.**

Commissioner & Secretary
Intermediate Education, Telangana.

CONTENTS

Unit **1**

FUNCTIONS

Functions: Let A and B be non-empty sets and f be a relation from A to B. If for each element $a \in$A, there exists a unique $b \in$B such that $(a, b) \in f$, then f is called a function (or) mapping from A to B. It is denoted by $f : A \to B$. The set A is called the domain of f and B is called the co-domain of f.

Range: If $f : A \to B$ is a function , then f(A), the set of all f-images of elements in A, is called the range of f. Clearly $f(A) = \{f(a) / a \in A\} \subseteq B$. Also f(A) = $\{b \in B / b = f(a)$ for some $a \in A\}$.

Injection or one-one function: A function $f : A \to B$ is called an injection if distinct elements of A have distinct f-images in B. An injection is also called a one-one function.

$f : A \to B$ is an injection $\Leftrightarrow a_1, a_2 \in$ A and $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$

$\Leftrightarrow a_1, a_2 \in$A and $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

Surjection: A function $f : A \to B$ is called a surjection if the range of f is equal to the co-domain.

$f : A \to B$ is a surjection $\Leftrightarrow$ range $f = f$(A) = B (co-domain)

$\Leftrightarrow$ B $= \{f(a) | a \in A\}$

$\Leftrightarrow$ for every $b \in$B there exists atleast one $a \in$A such that $f(a) = b$.

Bijection: If $f : A \to B$ is both an injection and surjection then f is said to be a bijection or one to one from A onto B.

(i.e.) $f : A \to B$ is a bijection $\Leftrightarrow f$ is both injection and surjection.

$\Leftrightarrow$ (i) If $a_1, a_2 \in$ A and $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

(ii) for every $b \in$ B $\exists$ atleast one $a \in$A such that $f(a) = b$.

Finite set: If A is empty or $\exists$ $n \in$ N such that there is a bijection from A onto {1,2,3..........., n} then A is called a finite set. In such a case we say that the number of elements in A is n and denote it by |A| or n(A).

Equality of functions: Let f and g be functions. We say f and g are equal and write $f = g$ if domain of f = domain of g and $f(x) = g(x)$ for all $x \in$ domain f.

Identity function: Let A be a non-empty set. Then the function $f: A \to A$ defined by $f(x) = x \ \forall \ x \in A$ is called the identity function on A and is denoted by I_A.

Constant function: A function $f: A \to B$ is said to be a constant function, if the range of f contains one and only one element i.e. $f(x) = c \ \forall \ x \in A$, for some fixed $c \in B$. In this case the constant function f will be denoted by c itself.

Very Short Answer Questions

1. If $f : R \setminus \{0\} \to R$ is defined by $f(x) = x + \frac{1}{x}$ then prove that $\left(f(x)\right)^2 = f(x^2) + f(1)$.

Sol. Since $f(x) = \left(x + \frac{1}{x}\right)$

$$f(x^2) + f(1) = x^2 + \frac{1}{x^2} + \left(1 + \frac{1}{1}\right)$$

$$= x^2 + \frac{1}{x^2} + 2$$

$$= \left(x + \frac{1}{x}\right)^2 = [f(x)]^2$$

2. If the function f is defined by $f(x) = \begin{cases} 3x - 2, x > 3 \\ x^2 - 2, -2 \le x \le 2 \\ 2x + 1, x < -3 \end{cases}$ then find the values, if exists, of

(i) $f(4)$, (ii) $f(2.5)$, (iii) $f(-2)$, (iv) $f(-4)$, (v) $f(0)$, (vi) $f(-7)$.

Sol. Note that the domain of f is $(-\infty, -3) \cup [-2, 2] \cup (3, \infty)$

(i) Since $f(x) = 3x - 2$, for $x > 3$, we have $f(4) = 12 - 2 = 10$

(ii) 2.5 does not belong to domain f, $f(2.5)$ is not defined.

(iii) Since $f(x) = x^2 - 2, -2 \le x \le 2$, we have $f(-2) = (-2)^2 - 2 = 2$

(iv) Since $f(x) = 2x + 1, x < -3$, we have $f(-4) = 2(-4) + 1 = -7$

(v) Since $f(x) = x^2 - 2, -2 \le x \le 2$, we have $f(0) = 0^2 - 2 = -2$

(vi) Since $f(x) = 2x + 1, x < -3$, we have $f(-7) = 2(-7) + 1 = -13$

3. If $A=\left\{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\right\}$ and $f: A \to B$ is surjection defined by $f(x) = Cos(x)$ then find B.

Sol. Let $f: A \to$ B be a surjection defined by $f(x) = Cos(x)$

Then B = rang of $f = f(A) = \left\{f(0), f\left(\frac{\pi}{6}\right), f\left(\frac{\pi}{4}\right), f\left(\frac{\pi}{3}\right), f\left(\frac{\pi}{2}\right)\right\}$

$$= \left[\text{Cos}0, \text{Cos}\frac{\pi}{6}, \text{Cos}\frac{\pi}{4}, \text{Cos}\frac{\pi}{3}, \text{Cos}\frac{\pi}{2}\right]$$

$$= \left\{1, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}, 0\right\}$$

4. If $f(x) = \frac{Cos^2x + Sin^4x}{Sin^2x + Cos^4x}$ $\forall\, x \in R$ then show that $f(2012) = 1$.

Sol. $f(x) = \frac{Cos^2x + Sin^4x}{Sin^2x + Cos^4x}$

$$= \frac{1 - Sin^2x + Sin^4x}{1 - Cos^2x + Cos^4x}$$

$$= \frac{1 - Sin^2x\ (1 - Sin^2x)}{1 - Cos^2x\ (1 - Cos^2x)}$$

$$= \frac{1 - Sin^2x\ \text{Cos}^2x}{1 - Sin^2xCos^2x}$$

$f(x) = 1$

$f(2012) = 1$

5. If the function f is defined by $f(x) = \begin{cases} x+2, & x > 1 \\ 2, & -1 \le x \le 1 \\ x-1, & -3<x<-1 \end{cases}$, then find the values of

(i) $f(3)$ (ii) $f(0)$ (iii) $f(-1.5)$ (iv) $f(2) + f(-2)$ (v) $f(-5)$

Sol. *(i)* Since $f(x) = x + 2, x > 1$ we have $f(3) = 3 + 2 = 5$

(ii) Since $f(x) = 2$, $-1 \le x \le 1$ we have $f(0) = 2$

(iii) Since $f(x) = x-1$, $-3<x<-1$ we have $f(-1.5) = -1.5-1 = -2.5$

(iv) Since $f(x) = x+2, x > 1$ we have $f(2) = 2+2 = 4$

$f(x) = x-1$, $-3<x<-1$ we have $f(-2) = -2-1 = -3$

$f(2) + f(-2) = 4 + (-3) = 4 - 3 = 1$

(v) As -5 does not belong to domain f, $f(-5)$ is not defined.

6. $f: R\setminus\{0\} \to R$ is defined by $f(x) = x^3 - \frac{1}{x^3}$ then show that $f(x) + f\left(\frac{1}{x}\right) = 0$.

Sol. $f(x) = x^3 - \frac{1}{x^3}$

$$f\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)^3 - \frac{1}{\left(\frac{1}{x}\right)^3}$$

$$f\left(\frac{1}{x}\right) = \frac{1}{x^3} - x^3$$

$$= f(x) + f\left(\frac{1}{x}\right) = x^3 - \frac{1}{x^3} + \frac{1}{x^3} - x^3 = 0$$

7. If $f: R \to R$ is defined by $f(x) = \frac{1-x^2}{1+x^2}$ then show that, $f(\tan\theta) = Cos2\theta$.

Sol. $f(x) = \frac{1-x^2}{1+x^2}$

$$f(\tan\theta) = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\frac{Sin^2\theta}{Cos^2\theta}}{1+\frac{Sin^2\theta}{Cos^2\theta}}$$

$$f(\tan\theta) = \frac{\frac{Cos^2\theta - Sin^2\theta}{Cos^2\theta}}{\frac{Cos^2\theta + Sin^2\theta}{Cos^2\theta}} = \frac{Cos^2\theta - Sin^2\theta}{Cos^2\theta + Sin^2\theta}$$

$$f(\tan\theta) = Cos^2\theta - Sin^2\theta = Cos2\theta$$

8. If $f: R\setminus[\pm 1] \to R$ is defined by $f(x) = \log\left|\frac{1+x}{1-x}\right|$, then show that $f\left(\frac{2x}{1+x^2}\right) = 2f(x)$.

Sol. $f(x) = \log\left|\frac{1+x}{1-x}\right|$

$$f\left(\frac{2x}{1+x^2}\right) = \log\left|\frac{1+\frac{2x}{1+x^2}}{1-\frac{2x}{1+x^2}}\right|$$

$$= \log\left|\frac{\frac{1+x^2+2x}{1+x^2}}{\frac{1+x^2-2x}{1+x^2}}\right|$$

$$= \log\left|\frac{(1+x)^2}{(1-x)^2}\right|$$

$$= \log\left|\frac{1+x}{1-x}\right|^2$$

$$= 2\log\left|\frac{1+x}{1-x}\right|$$

$$f\left(\frac{2x}{1+x^2}\right) = 2f(x)$$

9. A = {-2, -1, 0, 1, 2} and f : A $\to$ B is a surjection defined by $f(x) = x^2 + x + 1$ then find B.

Sol. f : A $\to$ B is a surjection $\Rightarrow \forall\ b \in B\ \exists\ a \in A$ such that $f(a) = b$

A = {-2, -1, 0, 1, 2}

$f(x) = x^2 + x + 1$

$f(-2) = (-2)^2 + (-2) + 1 \quad = 4 - 2 = 3$

$f(-1) = (-1)^2 + (-1) + 1 = 1$

$f(0) = (0)^2 + (0) + 1 = 1$

$f(1) = 1^2 + 1 + 1 = 3$

$f(2) = (2)^2 + 2 + 1 = 7$

$\therefore B = \{1, 3, 7\}$

10. A = {1, 2, 3, 4} and f : A $\to$ R is a function defined by $f(x) = \frac{x^2 - x + 1}{x + 1}$, then find the range of *f*.

Sol. f : A $\to$ R $\Rightarrow f$(A)=R

$$f(x) = \frac{x^2 - x + 1}{x + 1}$$

$$f(1) = \frac{1^2 - 1 + 1}{1 + 1} = \frac{1}{2}$$

$$f(2)=\frac{2^2-2+1}{2+1}=\frac{3}{3}=1$$

$$f(3)=\frac{3^2-3+1}{3+1}=\frac{7}{4}$$

$$f(4)=\frac{4^2-4+1}{4+1}=\frac{13}{5}$$

Range of f = $\left\{\frac{1}{2},1,\frac{7}{4},\frac{13}{5}\right\}$

11. If the function $f:\text{R}\to\text{R}$ defined by $f(x)=\frac{3^x+3^{-x}}{2}$, then show that $f(x+y)+f(x-y)=2f(x)f(y)$.

Sol. $f(x)=\frac{3^x+3^{-x}}{2}$, $f(y)=\frac{3^y+3^{-y}}{2}$

LHS $\Rightarrow f(x+y)+f(x-y)=\frac{3^{(x+y)}+3^{-(x+y)}}{2}+\frac{3^{x-y}+3^{-(x-y)}}{2}$

$$=\frac{1}{2}\left[3^{x+y}+3^{-(x+y)}+3^{(x-y)}+3^{-(x-y)}\right]$$

$$=\frac{1}{2}\left[3^x3^y+3^x3^{-y}+3^{-x}3^y+3^{-x}3^{-y}\right]$$

RHS $\Rightarrow 2f(x)f(y)$

$$=2\left(\frac{3^x+3^{-x}}{2}\right)\left(\frac{3^y+3^{-y}}{2}\right)$$

$$=\frac{1}{2}\left(3^x+3^{-x}\right)\left(3^y+3^{-y}\right)$$

$$=\frac{1}{2}\left(3^x3^y+3^x3^{-y}+3^{-x}3^y+3^{-x}3^{-y}\right)$$

$$=\frac{1}{2}\left(3^{x+y}+3^{x-y}+3^{-(x-y)}+3^{-(x+y)}\right)$$

$$=\frac{1}{2}\left(3^{(x+y)}+3^{-(x+y)}+3^{(x-y)}+3^{-(x-y)}\right)$$

LHS = RHS

$f(x+y)+f(x-y)=2f(x)f(y)$

PRACTICE PROBLEMS

1. $f : R \to R$ defined by $f(x) = \dfrac{4^x}{4^x + 2}$, then show that $f(1-x) = 1 - f(x)$ and hence deduce the value of $f\left(\dfrac{1}{4}\right) + 2f\left(\dfrac{1}{2}\right) + f\left(\dfrac{3}{4}\right)$.

Real valued function

If X is any set, $f : X \to R$ then *f* is called a real valued function.

12. Find the domains of the following real valued functions

i) $f(x) = \dfrac{1}{(x^2-1)(x+3)} \in R$

Sol. $\dfrac{1}{(x^2-1)(x+3)} \in R \Rightarrow \dfrac{1}{(x+1)(x-1)(x+3)} \in R$

$\Rightarrow (x+1)(x-1)(x+3) \neq 0 \Rightarrow x \neq -1, 1, -3,$

Domain of *f* is R \{1, -1, -3}

ii) $f(x) = \dfrac{2x^2 - 5x + 7}{(x-1)(x-2)(x-3)}$

Sol. $\dfrac{2x^2 - 5x + 7}{(x-1)(x-2)(x-3)} \in R$

$\Rightarrow (x-1)(x-2)(x-3) \neq 0$

$\Rightarrow x \neq 1, 2, 3$

Domain of *f* is $R \setminus \{1, 2, 3\}$

iii) $f(x) = \dfrac{1}{\log(2-x)}$

Sol. $\dfrac{1}{\log(2-x)} \in R$

$\Rightarrow (2-x) > 0$	$2 - x \neq 1$
$x - 2 < 0$	$-x \neq -1$
$x < 2$	$x \neq 1$

Domain of *f* is $(-\infty, 2) - \{1\}$

iv) $f(x)=|x-3|$

Sol. $f(x)=|x-3|$

$$f(x)=|x|=\begin{cases} x \text{ if } x\geq 0 \\ -x \text{ if } x<0 \end{cases}$$

$$\Rightarrow |x-3|=\begin{cases} x-3 \text{ if } x-3\geq 0 \\ -(x-3) \text{ if } x-3<0 \end{cases}$$

Domain of f is R.

v) $f(x)=\sqrt{4x-x^2}$

Sol. $f(x)=\sqrt{4x-x^2}\in R$

$\Leftrightarrow 4x-x^2\geq 0$

$\Leftrightarrow x^2-4x\leq 0$

$\Leftrightarrow x(x-4)\leq 0$

$\Leftrightarrow x\geq 0,\ x-4\leq 0$

Domain of f is [0, 4]

vi) $f(x)=\dfrac{1}{\sqrt{1-x^2}}$

Sol. $f(x)=\dfrac{1}{\sqrt{1-x^2}}$

$\sqrt{1-x^2}>0$

$\Rightarrow x^2-1<0$

$\Rightarrow (x+1)(x-1)<0$

$\Rightarrow (x+1)>0, (x-1)<0$

$x>-1; x<1$

Domain of f is (–1, 1)

13. Find the range of the following real valued functions

i) $\log|4-x^2|$

Sol. $f(x)=\log|4-x^2|$

$f(x)=\log x$; Range = $(-\infty,\infty)$

$f(x)=|x|$; Range = $[0,\infty)$

$f(x)\in R \Rightarrow 4-x^2\neq 0,\ x^2\neq 4,\ x\neq -2, 2$

Domain of f = R − {−2, 2}

Range of f = R

ii) $f(x)=\sqrt{[x]-x}$

Sol. $f(x)=\sqrt{[x]-x}$

$f(x)=[x]-x\geq 0$

$=[x]\geq x$

Domain of f = Integers Z

Range of f = {0}

iii) $f(x)=\dfrac{Sin\pi[x]}{1+[x^2]}$

Sol. $f(x)=\dfrac{Sin\pi[x]}{1+[x^2]}$

$=1+[x^2]\neq 0$

Domain of f = R $[\because Sin\pi=0]$

Range of f = {0}

iv) $f(x)=\dfrac{x^2-4}{x-2}$

Sol. $f(x)=\dfrac{x^2-4}{x-2}$

$x-2\neq 0$

$f(x)=\dfrac{x^2-4}{x-2}$

$=\dfrac{(x+2)(x-2)}{(x-2)}=x+2$

$f(x)\neq 2+2=4$

Domain of f = R – {2}

Range of f = R – {4}

PRACTICE PROBLEMS

I. Find the domains of the following real valued functions

(i) $f(x)=\dfrac{3^x}{x+1}$ Ans: R – {–1}

(ii) $f(x)=\sqrt{x^2-25}$ Ans: R – (–5, 5)

(iii) $f(x)=\sqrt{x-[x]}$ Ans: R

(iv) $f(x)=\sqrt{[x]-x}$ Ans: Z

(v) $f(x)=\dfrac{1}{6x-x^2+5}$ Ans: R – {1, 5}

(vi) $f(x)=\dfrac{1}{\sqrt{x^2-a^2}}(a>0)$ Ans: R – [–a, a]

(vii) $f(x)=\sqrt{(x+2)(x-3)}$ Ans: R – (–2, 3)

(viii) $f(x)=\sqrt{(x-\alpha)(\beta-x)}$ $(0<\alpha<\beta)$ Ans: $x\in[\alpha,\beta]$

(ix) $f(x)=\sqrt{2-x}+\sqrt{1+x}$ Ans: [–1, 2]

(x) $f(x)=\sqrt{x^2-1}+\dfrac{1}{\sqrt{x^2-3x+2}}$ Ans: R – [–1, 2]

II. Find the ranges of the following real valued functions

(i) $\sqrt{9+x^2}$ Ans: $[3,\infty)$

Long Answer Questions

1. If f = {(4,5), (5,6), (6,-4)}, g = {(4,-4), (6,5), (8,5)} then find

(i) $f+g$ (ii) $f-g$ (iii) $2f+4g$ (iv) $f+4$ (v) fg

(vi) f/g (vii) $|f|$ (viii) $\sqrt{f}$ (ix) f^2 (x) f^3

Sol. Domain of f = A = {4, 5, 6}

Domain of g = B = {4, 6, 8}

Domain of $f\pm g$ = $A\cap B$ = [4, 6]

(i) $f+g=\{(4,5-4),(6,-4+5)\}=\{(4,1),\ (6,1)\}$

(ii) $f-g=\{(4,5+4),(6,-4-5)\}=\{(4,9),(6,-9)\}$

(iii) Domain of $2f$ = A = {4, 5, 6}

Domain of 4 g = B = {4, 6, 8}

$\therefore\ 2f=\{(4,10),(5,12),(6,-8)\}$

$\therefore\ 4g=\{(4,-16),\ (6,20),\ (8,20)\}$

Domain of $2f+4g$ = {4, 6}

$2f+4g$ = {(4, 10–16), (6, –8+20)} = {(4, –6), (6, 12)}

(iv) Domain of $f + 4$ = A = {4, 5, 6}

$f + 4$ = { (4, 5+4), (5, 6+4), (6, –4+4)}

= {(4, 9), (5, 10), (6,0)}

(v) Domain of fg = $A \cap B$ = {4, 6}

fg = { (4, (5)(–4)), (6, (–4)(–5))}

= {(4, –20), (6, 20)}

(vi) Domain of $\frac{f}{g}$ = {4, 6}

$$\therefore \frac{f}{g} = \left\{\left(4, \frac{-5}{4}\right), \left(6, \frac{-4}{5}\right)\right\}$$

(vii) Domain of | f | A = {4, 5, 6}

$|f|$ = {(4, 5), (5, 6), (6, 4)}

(viii) Domain of $\sqrt{f}$ {4,5}

$$\sqrt{f} = \{(4, \sqrt{5}), (5, \sqrt{6})\}$$

(ix) Domain of f^2 = A = {4, 5, 6}

$$f^2 = \{(4, 25), (5, 36), (6, 16)\}$$

(x) Domain of f^3 = A = {4, 5, 6}

$$f^3 = \{(4, 125), (5, 216), (6, -64)\}$$

2. If $f(x) = x^2$ and $g(x) = |x|$, find the following functions:

(i) $f + g$ (ii) $f - g$ (iii) fg (iv) $2f$ (v) f^2 (vi) $f + 3$

Sol. $f(x) = x^2$

$$g(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Domain f = Domain of g = R

Hence the domain of all the functions is R.

(i) $(f + g)(x) = f(x) + g(x) = x^2 + |x| = \begin{cases} x^2 + x, & x \geq 0 \\ x^2 - x, & x < 0 \end{cases}$

(ii) $(f - g)(x) = f(x) - g(x) = x^2 - |x| = \begin{cases} x^2 - x, & x > 0 \\ x^2 + x, & x < 0 \end{cases}$

(iii) $(fg)(x) = f(x)g(x) = x^2|x| = \begin{cases} x^3, & x \geq 0 \\ -x^3, & x < 0 \end{cases}$

(iv) $(2f)\,x = 2f(x) = 2x^2$

(v) $f^2(x) = (f(x))^2 = (x^2)^2 = x^4$

(vi) $(f+3)(x) = f(x) + 3 = x^2 + 3$

3. If f and g are real valued functions defined by $f(x) = 2x-1$, $g(x) = x^2$, then find

(i) $(3f-2g)\,(x)$ (ii) $(fg)(x)$ (iii) $\left(\frac{\sqrt{f}}{g}\right)(x)$ (iv) $(f+g+2)\,(x)$

Sol. $f(x) = 2x-1$, $g(x) = x^2$

$\Rightarrow (f-g)x = f(x) - g(x)$

(i) $(3f-2g)(x) = 3f(x) - 2g(x) = 3(2x\text{-}1)\ \text{-}\ 2(x^2)$

$= 6x - 3 - 2x^2$

$= -2x^2 + 6x - 3$

$(3f-2g)\,x = -2x^2 + 6x - 3$

(ii) $(fg)(x) = f(x).g(x)$

$= (2x-1)(x^2) = 2x^3 - x^2$

(iii) $\left(\frac{\sqrt{f}}{g}\right)x = \frac{\sqrt{f(x)}}{g(x)} = \frac{\sqrt{2x-1}}{x^2}$

(iv) $(f+g+2)x = f(x) + g(x) + 2$

$= 2x - 1 + x^2 + 2$

$= x^2 + 2x + 1 = (x+1)^2$

4. If $f = \{(1,2),(2,-3),(3,-1)\}$ then find

(i) $2f$ (ii) $2+f$ (iii) $\sqrt{f}$ (iv) f^2

Sol. $f = \{(1,2),(2,-3),(3,-1)\}$

Domain of f, A = {1, 2, 3}

(i) $2f = \{(1, 2\times2), (2, 2(-3)), (3, 2(-1)\} = \{(1,4), (2,-6), (3,-2)\}$

(ii) $2 + f = \{(1, 2+2), (2, -3+2), (3, -1+2)\}$

$2 + f = \{(1,4), (2,-1), (3,1)\}$

(iii) $\sqrt{f} = \{(1, \sqrt{2})\}$

(iv) $f^2 = \{(1, 2^2), (2,(-3)^2),(3,(-1)^2)\} = \{(1,4), (2,9), (3,1)\}$

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Unit 3

MATRICES

Matrix

An ordered rectangular array of elements is called as matrix.

Ex: $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 0 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$

Order of Matrix

A matrix having m rows and n columns is said to be of order m × n, read as m cross n or m by n.

Types of Matrices

1. **Square Matrix:** A matrix in which the number of rows is equal to the number of columns is called a square matrix.

Ex: $\begin{bmatrix} 1 & -1 \\ 0 & 4 \end{bmatrix}_{2\times 2} \quad \begin{bmatrix} 2 & 0 & 1 \\ 4 & -1 & 2 \\ 7 & 6 & 9 \end{bmatrix}_{3\times 3}$

Principal Diagonal / Diagonal

If $A = [a_{ij}]$ is a square matrix of order n, the elements a_{11}, a_{12} ...., a_{nn} are said to constitute its principal diagonal or simply the diagonal. Hence a_{ij} is an element of the diagonal according as i = j.

$$\begin{bmatrix} 2 & 0 & 1 \\ 4 & -1 & 2 \\ 7 & 6 & 9 \end{bmatrix}$$

Trace of Matrix

The sum of the elements of the diagonal of a square matrix A is called the trace of A and is denoted by Tr(A).

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} a_{ii}$$

Ex: If $A = \begin{bmatrix} 2 & 0 & 1 \\ 4 & -1 & 2 \\ 7 & 6 & 9 \end{bmatrix}$ then $\operatorname{Tr}(A) = 2 + (-1) + 9 = 10$

2. Diagonal Matrix

If each non-diagonal element of a square matrix is equal to zero, then the matrix is called a diagonal matrix.

Ex: $\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are diagonal matrices.

3. Scalar Matrix

If each non-diagonal element of a square matrix is zero and all diagonal elements are equal to each other, then it is called a scalar matrix.

Ex: $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ are all scalar matrices.

4. Unit matrx / Identity matrix

If each non-diagonal element of a square matrix is equal to zero and each diagonal element is equal to 1, then that matrix is called a unit matrix or identity matrix.

Ex: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$ are identity matrices.

5. Null Matrix or Zero matrix

If each element of a matrix is zero, then it is called a null matrix or zero matrix. It is denoted by $O_{m\times n}$ or O.

Ex.: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}, \; O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{3\times 2}$

6. Row matrix

A matrix with only one row is called a Row matrix.

Ex: $\begin{bmatrix} 1 & 3 & -2 \end{bmatrix}_{1\times 3}$

7. Column Matrix

A matrix with only one column is called a column matrix.

Ex: $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}_{3\times 1}$

8. Triangular matrices

A square matrix $A = [a_{ij}]$ is said to be upper triangular if $a_{ij} = 0$ for all $i > j$.

'A' is said to be lower triangular if $a_{ij} = 0 \ \forall \ i < j$.

Ex: $\begin{bmatrix} 2 & -4 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -3 & 1 \\ 0 & 4 \end{bmatrix}$ are upper triangular matrices.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ are lower triangular matrices.

Equality of matrices

Matrices A and B are said to be equal if A and B are of the same order and the corresponding elements of A and B are the same.

Thus $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$

are equal if $a_{ij} = b_{ij}$ for $i = 1, 2, 3$ and $j = 1, 2, 3$

Sum of two matrices

Let A and B be matrices of the same order. Then the sum of A and B, denoted by A + B is defined as the matrix of the same order in which each element is the sum of the corresponding elements of A and B.

Scalar multiple of a matrix

Let A be a matrix of order $m \times n$ and k be a scalar. Then the $m \times n$ matrix obtained by multiplying each element of A by k is called a scalar multiple of A and is denoted by kA.

If $A = [a_{ij}]_{m\times n}$ then $kA = [ka_{ij}]_{m\times n}$

Properties of Scalar multiplication of a matrix

Let A and B be matrices of the same order and α, β be scalars. Then

(i) $\alpha(\beta A) = (\alpha\beta)A = \beta(\alpha A)$

(ii) $(\alpha+\beta)A = \alpha A + \beta A$

(iii) $0A = O$

(iv) $\alpha O = O$

(v) $\alpha(A+B) = \alpha A + \alpha B$

Very Short Answer Questions

1. If $A = \begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 3 & -5 \end{bmatrix}$, $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ and A + B = X then find the values of x_1, x_2, x_3, x_4 .

Sol. $A = \begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 3 & -5 \end{bmatrix}, X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$

$A + B = X$

$$\Rightarrow \begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ 7 & -3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \Rightarrow \begin{matrix} x_1 = 1 \\ x_2 = 4 \\ x_3 = 7 \\ x_4 = -3 \end{matrix}$$

2. $A = \begin{bmatrix} -1 & -2 & 3 \\ 1 & 2 & 4 \\ 2 & -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & -2 & 5 \\ 1 & -2 & 2 \\ 1 & 2 & -3 \end{bmatrix}, C = \begin{bmatrix} -2 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$ then find A + B + C.

Sol. $A = \begin{bmatrix} -1 & -2 & 3 \\ 1 & 2 & 4 \\ 2 & -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & -2 & 5 \\ 1 & -2 & 2 \\ 1 & 2 & -3 \end{bmatrix}, C = \begin{bmatrix} -2 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$

$$A + B + C = \begin{bmatrix} -1+1+(-2) & -2+(-2)+1 & 3+5+2 \\ 1+1+1 & 2+(-2)+1 & 4+2+2 \\ 2+1+2 & -1+2+0 & 3+(-3)+1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -3 & 10 \\ 3 & 1 & 8 \\ 5 & 1 & 1 \end{bmatrix}$$

3. If $A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & -2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & -1 & 0 \\ 2 & 1 & 3 \\ 4 & -1 & 2 \end{bmatrix}$ and X = A + B then find X.

Sol. X = A + B

$$X = \begin{bmatrix} 3 & 2 & -1 \\ 2 & -2 & 0 \\ 1 & 3 & 1 \end{bmatrix} + \begin{bmatrix} -3 & -1 & 0 \\ 2 & 1 & 3 \\ 4 & -1 & 2 \end{bmatrix}$$

$$X = \begin{bmatrix} 3+(-3) & 2+(-1) & -1+0 \\ 2+2 & -2+1 & 0+3 \\ 1+4 & 3+(-1) & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -1 & 3 \\ 5 & 2 & 3 \end{bmatrix}$$

4. If $\begin{bmatrix} x-3 & 2y-8 \\ z+2 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ -2 & a-4 \end{bmatrix}$ then find the values of x, y, z, a.

Sol. $\begin{bmatrix} x-3 & 2y-8 \\ z+2 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ -2 & a-4 \end{bmatrix}$

$\Rightarrow x - 3 = 5 \Rightarrow x = 5 + 3 = 8 \Rightarrow \boxed{x = 8}$

$\Rightarrow 2y - 8 = 2 \Rightarrow 2y = 2 + 8 = 10$

$2y = 10$

$y = \frac{10}{2} = 5 \Rightarrow \boxed{y = 5}$

$\Rightarrow z + 2 = -2 \Rightarrow z = -2-2 = -4 \quad \boxed{z = -4}$

$\Rightarrow 6 = a - 4 \Rightarrow a = 6 + 4 \Rightarrow \boxed{a = 10}$

5. If $\begin{bmatrix} x-1 & 2 & 5-y \\ 0 & z-1 & 7 \\ 1 & 0 & a-5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 1 & 0 & 0 \end{bmatrix}$ then find the values of x, y, z, a.

Sol. $\begin{bmatrix} x-1 & 2 & 5-y \\ 0 & z-1 & 7 \\ 1 & 0 & a-5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 7 \\ 1 & 0 & 0 \end{bmatrix}$

$\Rightarrow x - 1 = 1 \Rightarrow x = 1 + 1 = 2 \Rightarrow \boxed{x = 2}$

$\Rightarrow 5 - y = 3 \Rightarrow y = 5 - 3 = 2 \Rightarrow \boxed{y = 2}$

$\Rightarrow z - 1 = 4 \Rightarrow z = 4 + 1 = 5 \Rightarrow \boxed{z = 5}$

$\Rightarrow a - 5 = 0 \Rightarrow \boxed{a = 5}$

6. Find the trace of $A = \begin{bmatrix} 1 & 3 & -5 \\ 2 & -1 & 5 \\ 2 & 0 & 1 \end{bmatrix}$.

Sol. $A = \begin{bmatrix} 1 & 3 & -5 \\ 2 & -1 & 5 \\ 2 & 0 & 1 \end{bmatrix}$

Trace of A = 1 + (–1) + 1 = 1

7. If $A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ then find B – A and 4A – 5B.

Sol. $A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & -6 \end{bmatrix}, B = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$$\Rightarrow B - A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & -6 \end{bmatrix}$$

$$B - A = \begin{bmatrix} -1 & 1 & 1 \\ -2 & -2 & -4 \\ -4 & -5 & 5 \end{bmatrix}$$

$$4A - 5B = 4\begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & -6 \end{bmatrix} - 5\begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 4 & 8 \\ 8 & 12 & 16 \\ 16 & 20 & -24 \end{bmatrix} - \begin{bmatrix} -5 & 10 & 15 \\ 0 & 5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$4A - 5B = \begin{bmatrix} 5 & -6 & -7 \\ 8 & 7 & 16 \\ 16 & 20 & -19 \end{bmatrix}$$

8. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ then find 3B – 2A.

Sol. $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \Rightarrow 2A = \begin{bmatrix} 2 & 4 & 6 \\ 6 & 4 & 2 \end{bmatrix}$

$B = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \Rightarrow 3B = \begin{bmatrix} 9 & 6 & 3 \\ 3 & 6 & 9 \end{bmatrix}$

$3B - 2A = \begin{bmatrix} 9 & 6 & 3 \\ 3 & 6 & 9 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 6 \\ 6 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 2 & -3 \\ -3 & 2 & 7 \end{bmatrix}$

9. If $A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ then show that $A^2 = -I$, $(i^2 = -1)$.

Sol. $A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

$A \times A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \times \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

$A^2 = \begin{bmatrix} i^2 + 0 & 0 + 0 \\ 0 + 0 & 0 + i^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$A^2 = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \boxed{A^2 = -I}$

10. If $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$ then find A^2.

Sol. $A^2 = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} \times \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 16 + (-2) & 8 + 2 \\ -4 + (-1) & -2 + 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 14 & 10 \\ -5 & -1 \end{bmatrix}$$

11. If $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ then find A^2.

Sol. $A^2 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \times \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$

$$A^2 = \begin{bmatrix} i^2 + 0 & 0 + 0 \\ 0 + 0 & 0 + i^2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \boxed{A^2 = -I}$$

12. If $A = \begin{bmatrix} 2 & 4 \\ -1 & k \end{bmatrix}$ and $A^2 = O$ then find k.

Sol. $A = \begin{bmatrix} 2 & 4 \\ -1 & k \end{bmatrix}$

$$A^2 = A \times A = \begin{bmatrix} 2 & 4 \\ -1 & k \end{bmatrix} \times \begin{bmatrix} 2 & 4 \\ -1 & k \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 4 + (-4) & 8 + 4k \\ -2 + (-k) & -4 + k^2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 8 + 4k \\ -2 - k & -4 + k^2 \end{bmatrix}$$

$A^2 = O$

$$\begin{bmatrix} 0 & 8 + 4k \\ -2 - k & -4 + k^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$8 + 4k = 0$

$4k = -8$

$k = \frac{-8}{4} = -2$

$\therefore \; k = -2$

13. If $A = \begin{bmatrix} -2 & 1 \\ 5 & 0 \\ -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 3 & 1 \\ 4 & 0 & 2 \end{bmatrix}$ then find $2A+B^1$ and $3B^1-A$.

Sol. $A = \begin{bmatrix} -2 & 1 \\ 5 & 0 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 3 & 1 \\ 4 & 0 & 2 \end{bmatrix}$

$$2A = 2\begin{bmatrix} -2 & 1 \\ 5 & 0 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 10 & 0 \\ -2 & 8 \end{bmatrix}$$

$$B = \begin{bmatrix} -2 & 3 & 1 \\ 4 & 0 & 2 \end{bmatrix} \; B^1 = \begin{bmatrix} -2 & 4 \\ 3 & 0 \\ 1 & 2 \end{bmatrix}$$

$$\Rightarrow 2A + B^1 = \begin{bmatrix} -4 & 2 \\ 10 & 0 \\ -2 & 8 \end{bmatrix} + \begin{bmatrix} -2 & 4 \\ 3 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 13 & 0 \\ -1 & 10 \end{bmatrix}$$

$$\Rightarrow 3B^1 = 3\begin{bmatrix} -2 & 4 \\ 3 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -6 & 12 \\ 9 & 0 \\ 3 & 6 \end{bmatrix}$$

$$\Rightarrow 3B^1 - A = \begin{bmatrix} -6 & 12 \\ 9 & 0 \\ 3 & 6 \end{bmatrix} - \begin{bmatrix} -2 & 1 \\ 5 & 0 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 11 \\ 4 & 0 \\ 4 & 2 \end{bmatrix}$$

14. If $A = \begin{bmatrix} 2 & -4 \\ -5 & 3 \end{bmatrix}$ then find $A+A^1$, AA^1.

Sol. $A = \begin{bmatrix} 2 & -4 \\ -5 & 3 \end{bmatrix}$, $A^1 = \begin{bmatrix} 2 & -5 \\ -4 & 3 \end{bmatrix}$

$$A+A^1 = \begin{bmatrix} 2 & -4 \\ -5 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -9 \\ -9 & 6 \end{bmatrix}$$

$$AA^1 = \begin{bmatrix} 2 & -4 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ -4 & 3 \end{bmatrix}$$

$$AA^1 = \begin{bmatrix} 4+16 & -10-12 \\ -10-12 & 25+9 \end{bmatrix}$$

$$AA^1 = \begin{bmatrix} 20 & -22 \\ -22 & 34 \end{bmatrix}$$

Symmetric matrix

A square matrix A is said to be symmetric if $A^1 = A$.

$$\text{Ex: } A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -3 & -1 \\ 0 & -1 & 4 \end{bmatrix}$$

Skew Symmetric matrix

A square matrix A is said to be skew symmetric if $A^1 = -A$.

$$\text{Ex: } \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 4 \\ 2 & -4 & 0 \end{bmatrix}$$

15. If $A = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & x & 7 \end{bmatrix}$ is a symmetric matrix, then find x.

Sol. $A = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & x & 7 \end{bmatrix}$, $A^1 = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 5 & x \\ 3 & 6 & 7 \end{bmatrix}$

A is Symmetric matrix $\Leftrightarrow \boxed{A^1 = A}$

$$\begin{bmatrix} -1 & 2 & 3 \\ 2 & 5 & x \\ 3 & 6 & 7 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & x & 7 \end{bmatrix}$$

$\Leftrightarrow \boxed{x = 6}$

16. If $A = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -2 \\ -1 & x & 0 \end{bmatrix}$ is a skew symmetric matrix, then find x.

Sol. A is Skew symmetric matrix $\Leftrightarrow \boxed{A^1 = -A}$

$$\begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & x \\ 1 & -2 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -2 \\ -1 & x & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & x \\ 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & 2 \\ 1 & -x & 0 \end{bmatrix}$$

$$\Rightarrow \boxed{x = 2}$$

17. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ then show that $AA^1 = A^1A = I$.

Sol. $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, $A^1 = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$$AA^1 = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & -\cos\alpha\ \sin\alpha + \sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha + \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$A^1A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \cdot \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & -\cos\alpha\ \sin\alpha + \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha - \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\boxed{\therefore\quad AA^1 = A^1A = I}$$

Short Answer Questions (4 marks)

1. If $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}$ then find AB, BA.

Sol. $AB = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}_{3\times3} \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}_{3\times2} = \begin{bmatrix} 0-1+4 & 0+0-2 \\ 1-2+6 & -2+0-3 \\ 2-3+8 & -4+0-4 \end{bmatrix}$

$\Rightarrow AB = \begin{bmatrix} 3 & -2 \\ 5 & -5 \\ 7 & -8 \end{bmatrix}$

$\Rightarrow BA = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}_{3\times2} \times \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}_{3\times3}$

Since the number of column of B is not equal to number of rows of A, BA is not defined.

2. If $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$ then examine whether A and B commute with respect to multiplication of matrices.

Sol. Both A and B are square matrices of order 3. Hence both AB, BA are defined and are matrices of order 3.

$AB = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}_{3\times3} \times \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}_{3\times3}$

$= \begin{bmatrix} 1+0+3 & 0-2+6 & 2-4+0 \\ 2+0-1 & 0+3-2 & 4+6+0 \\ -3+0+2 & 0+1+4 & -6+2+0 \end{bmatrix}$

$= \begin{bmatrix} 4 & 4 & -2 \\ 1 & 1 & 10 \\ -1 & 5 & -4 \end{bmatrix}$

$$BA = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0-6 & -2+0+2 & 3+0+4 \\ 0+2-6 & 0+3+2 & 0-1+4 \\ 1+4+0 & -2+6+0 & 3-2+0 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 0 & 7 \\ -4 & 5 & 3 \\ 5 & 4 & 1 \end{bmatrix}_{3\times 3}$$

Which shows that AB ≠ BA.

∴ A and B do not commute with respect to multiplication of matrices.

3. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}_{3\times 3}$ then show that $A^2 - 4A - 5I = O$.

Sol. $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$$\Rightarrow A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$\Rightarrow 4A = 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}$$

$$\Rightarrow 5I = 5\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}5 & 0 & 0\\0 & 5 & 0\\0 & 0 & 5\end{bmatrix}$$

Hence $A^2 - 4A - 5I$

$$= \begin{bmatrix}9 & 8 & 8\\8 & 9 & 8\\8 & 8 & 9\end{bmatrix} - \begin{bmatrix}4 & 8 & 8\\8 & 4 & 8\\8 & 8 & 4\end{bmatrix} - \begin{bmatrix}5 & 0 & 0\\0 & 5 & 0\\0 & 0 & 5\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

$$\Rightarrow A^2 - 4A - 5I = O$$

4. If $A = \begin{bmatrix}1 & -2 & 3\\-4 & 2 & 5\end{bmatrix}$ and $B = \begin{bmatrix}2 & 3\\4 & 5\\2 & 1\end{bmatrix}$ then do AB and BA exist? If they exist, find them. Do A and B commute with respect to multiplication?

Sol. $A = \begin{bmatrix}1 & -2 & 3\\-4 & 2 & 5\end{bmatrix}_{2\times3}$, $B = \begin{bmatrix}2 & 3\\4 & 5\\2 & 1\end{bmatrix}_{3\times2}$

AB multiplication matrix is 2×2 matrix

BA multiplication matrix is 3×3 matrix

$$\therefore AB = \begin{bmatrix}1 & -2 & 3\\-4 & 2 & 5\end{bmatrix} \times \begin{bmatrix}2 & 3\\4 & 5\\2 & 1\end{bmatrix}$$

$$AB = \begin{bmatrix}2-8+6 & 3-10+3\\-8+8+10 & -12+10+5\end{bmatrix} = \begin{bmatrix}0 & -4\\10 & 3\end{bmatrix}$$

$$BA = \begin{bmatrix}2 & 3\\4 & 5\\2 & 1\end{bmatrix} \times \begin{bmatrix}1 & -2 & 3\\-4 & 2 & 5\end{bmatrix}$$

$$= \begin{bmatrix}2-12 & -4+6 & 6+15\\4-20 & -8+10 & 12+25\\2-4 & -4+2 & 6+5\end{bmatrix} = \begin{bmatrix}-10 & 2 & 21\\-16 & 2 & 37\\-2 & -2 & 11\end{bmatrix}$$

Since $AB \neq BA$, A and B are not commutative with respect to multiplication.

5. If $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ then find A^4.

Sol. $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\Rightarrow A^4 = \left\{3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}^4 = 3^4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^4$$

$$A^4 = 81\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix}$$

6. If $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ then find A^3.

Sol. $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+5-6 & 1+2-3 & 3+6-9 \\ 5+10-12 & 5+4-6 & 15+12-18 \\ -2-5+6 & -2-2+3 & -6-6+9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0+0+0 & 0+0+0 & 0+0+0 \\ 3+15-18 & 3+6-9 & 9+18-27 \\ -1-5+6 & -1-2+3 & -3-6+9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

7. If $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ then find A^3-3A^2-A-3I (where I is unit matrix of order 3).

Sol. $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$

$$A^2 = A \times A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1+0+3 & -2-2-1 & 1+2+1 \\ 0+0-3 & 0+1+1 & 0-1-1 \\ 3+0+3 & -6-1-1 & 3+1+1 \end{bmatrix} = \begin{bmatrix} 4 & -5 & 4 \\ -3 & 2 & -2 \\ 6 & -8 & 5 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} 4 & -5 & 4 \\ -3 & 2 & -2 \\ 6 & -8 & 5 \end{bmatrix} \times \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 4+0+12 & -8-5-4 & 4+5+4 \\ -3+0-6 & 6+2+2 & -3-2-2 \\ 6+0+15 & -12-8-5 & 6+8+5 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 16 & -17 & 13 \\ -9 & 10 & -7 \\ 21 & -25 & 19 \end{bmatrix}$$

$$3A^2 = 3\begin{bmatrix} 4 & -5 & 4 \\ -3 & 2 & -2 \\ 6 & -8 & 5 \end{bmatrix} = \begin{bmatrix} 12 & -15 & 12 \\ -9 & 6 & -6 \\ 18 & -24 & 15 \end{bmatrix}$$

$$3I = 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$\Rightarrow A^3-3A^2-A-3I$

$$= \begin{bmatrix} 16 & -17 & 13 \\ -9 & 10 & -7 \\ 21 & -25 & 19 \end{bmatrix} - \begin{bmatrix} 12 & -15 & 12 \\ -9 & 6 & -6 \\ 18 & -24 & 15 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3} = O$$

$\therefore$ $\boxed{A^3 - 3A^2 - A - 3I = O}$

8. If $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ then show that $(aI+bE)^3 = a^3I + 3a^2bE$. (Where I is unit matrix).

Sol. LHS = $(aI + bE)^3$

$$= \left[a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right]^3$$

$$= \left[\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}\right]^3 = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}^3$$

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}^2 = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \times \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^2+0 & ab+ba \\ 0+0 & 0+a^2 \end{bmatrix} = \begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}^3 = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}^2 \times \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix} \times \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

$$\text{L.H.S.} = \begin{bmatrix} a^3+0 & a^2b+2a^2b \\ 0+0 & 0+a^3 \end{bmatrix} = \begin{bmatrix} a^2 & 3a^2b \\ 0 & a^3 \end{bmatrix}$$

R.H.S. = $a^3I + 3a^2bE$

$$= a^3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 3a^2b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} a^3 & 0 \\ 0 & a^3 \end{bmatrix} + \begin{bmatrix} 0 & 3a^2b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{bmatrix}$$

$\therefore$ L.H.S. = RHS

$$\boxed{(aI+bE)^3 = a^3I + 3a^2bE}$$

9. If $\theta - \phi = \frac{\pi}{2}$, then show that $\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix} = O.$

Sol. $\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\theta\cos^2\phi + \cos\theta\sin\theta\cdot\cos\phi\sin\phi & \cos^2\theta\cdot\cos\phi\sin\phi + \cos\theta\sin\theta\cdot\sin^2\phi \\ \cos\theta\sin\theta.\cos^2\phi + \sin^2\theta\cdot\cos\phi\sin\phi & \cos\theta\sin\theta\cdot\cos\phi\sin\phi + \sin^2\theta\cdot\sin^2\phi \end{bmatrix}$$

$$\theta - \phi = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{2} + \phi$$

$$\Rightarrow \cos\theta = \cos\left(\frac{\pi}{2}+\phi\right) = -\sin\phi$$

$$\sin\theta = \sin\left(\frac{\pi}{2}+\phi\right) = \cos\phi$$

$$= \begin{bmatrix} \sin^2\phi\cos^2\phi - \sin^2\phi\cos^2\phi & \sin^2\phi\cos\phi\sin\phi - \sin\phi\cos\phi\sin^2\phi \\ (-\sin\phi\cos\phi)\left(\cos^2\phi\right) + \cos^2\phi\cos\phi\sin\phi & -\sin\phi\cos\phi\cos\phi\sin\phi + \cos^2\phi\sin^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} \sin^2\phi\cos^2\phi - \sin^2\phi\cos^2\phi & \sin^3\phi\cos\phi - \sin^3\phi\cos\phi \\ -\sin\phi\cos^3\phi + \sin\phi\cos^3\phi & -\sin^2\phi\cos^2\phi + \sin^2\phi\cos^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

= O.

Singular Matrix

A square matrix is said to be singular if its determinant is zero.

Non-singular Matrix

A square matrix is said to be non-singular if its determinant is non-zero.

Adjoint of a matrix

The transpose of the matrix formed by replacing the elements of a square matrix A, with the corresponding co-factors is called the adjoint of A and is denoted by Adj A.

Invertible Matrix

Let A be a square matrix, we say that A is invertible if a matrix B exists such that AB = BA = I, where I is the unit matrix of the same order as A and B.

10. If $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is a non-singular matrix then show that A is invertible and $A^{-1} = \dfrac{\text{Adj}\,A}{\det A}$.

Sol. $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$$\text{Adj}A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

$$A\,.\,\text{Adj}A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \times \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

$$\begin{bmatrix} a_1A_1 + b_1B_1 + c_1C_1 & a_1A_2 + b_1B_2 + c_1C_2 & a_1A_3 + b_1B_3 + c_1C_3 \\ a_2A_1 + b_2B_1 + c_2C_1 & a_2A_2 + b_2B_2 + c_2C_2 & a_2A_3 + b_2B_3 + c_2C_3 \\ a_3A_1 + b_3B_1 + c_3C_1 & a_3A_2 + b_3B_2 + c_3C_2 & a_3A_3 + b_3B_3 + c_3C_3 \end{bmatrix}$$

$$= \begin{bmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{bmatrix} = \det A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$= \det A\,.\,I$

Since $\det A \neq 0$,

$A.(\text{Adj}A) = \det A\,.\,I$

$$\Rightarrow A\left(\frac{\text{Adj}\,A}{\det A}\right) = I$$

Similarly $\left(\dfrac{\text{Adj}\,A}{\det A}\right) \cdot A = I$

Let B = $\dfrac{\text{Adj A}}{\det A}$ then AB = BA = I

Hence A is invertible and $A^{-1} = B = \dfrac{\text{Adj A}}{\det A}$

Long Answer Questions (7 Marks)

1. Find the adjoint and the inverse of the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$.

Sol. $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

$$\det A = 1\,(16 - 9) - 3\,(4 - 3) + 3\,(3 - 4)$$
$$= 7 - 3 - 3 = 1 \neq 0$$

$\therefore$ A is invertible.

The cofactor matrix of A is $B = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

$$\text{Adj } A = B^T = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj A}}{\det A} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad [\because \det A = 1]$$

2. Show that $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$ is non-singular and find A^{-1}.

Sol. $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$

det A $= 1\,(4-3) - 2\,(6-3) + 1\,(3-2)$

$= 1 - 6 + 1 = -4 \neq 0$

Hence A is a non-singular matrix.

The cofactor matrix of A is B = $\begin{bmatrix} 1 & -3 & 1 \\ -3 & 1 & 1 \\ 4 & 0 & -4 \end{bmatrix}$

$$\text{Adj } A = B^T = \begin{bmatrix} 1 & -3 & 4 \\ -3 & 1 & 0 \\ 1 & 1 & -4 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj}\,A}{\det A} = \frac{1}{-4}\begin{bmatrix} 1 & -3 & 4 \\ -3 & 1 & 0 \\ 1 & 1 & -4 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & -1 \\ \frac{3}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 1 \end{bmatrix}$$

3. If A = $\begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$ then find $(A')^{-1}$.

Sol. A = $\begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$

$$A^1 = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$$

del $(A^1) = 1\,(-1-8) + 0 - 2\,(-8+3)$

$= -9 + 0 + 10 = 1 \neq 0$

$$\text{Cofactor matrix of } A^1 = \begin{bmatrix} -9 & 8 & -5 \\ -8 & 7 & -4 \\ -2 & 2 & -1 \end{bmatrix}$$

$$\text{Adjoint matrix of } A^1 = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

$$(A^1)^{-1} = \frac{\text{Adj}(A^1)}{\det(A^1)} = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

4. If $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ then show that $\text{adj}A = 3A^1$. Find A^{-1}.

Sol. $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

$$A^1 = \begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

$$3A^1 = 3\begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

$$3A^1 = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix} \quad ....(1)$$

$$\text{Cofactor matrix of } A = \begin{bmatrix} -3 & -6 & -6 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix}$$

$$\text{Adj}A = \begin{bmatrix} -3 & -6 & -6 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix}^T = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix} \qquad ....(2)$$

From (1) and (2) **Adj A = 3A^1**

$\det A = -1\,(1-4) + 2\,(2+4) - 2\,(-4-2)$

$= 3 + 12 + 12 = 27 \neq 0$

$$A^{-1} = \frac{\text{Adj}\,A}{\det A} = \frac{1}{27}\begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -\frac{3}{27} & \frac{6}{27} & \frac{6}{27} \\ -\frac{6}{27} & \frac{3}{27} & -\frac{6}{27} \\ -\frac{6}{27} & \frac{-6}{27} & \frac{3}{27} \end{bmatrix} = \frac{1}{9}\begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

5. If $3A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$ then show that $A^{-1} = A^T$.

Sol. $3A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$

$$A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$$

$AA^T = I \Rightarrow A^{-1} = A^T$

$$A^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$A \times A^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix} \times \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1+4+4 & 2+2-4 & -2+4-2 \\ 2+2-4 & 4+1+4 & -4+2+2 \\ -2+4-2 & -4+2+2 & 4+4+1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$AA^T = I$

$\Rightarrow A^{-1} = A^T$

Solution of Simultaneous Linear Equations

Cramer's Rule

Consider the system of equations

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is non-singular matrix

Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the solution of the equation AX = D where $D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

$\Rightarrow$ Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Then $x\Delta = \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix}$

On applying $C_1 \to C_1 + yC_2 + zC_3$ we get

$$x\Delta = \begin{vmatrix} a_1x+b_1y+c_1z & b_1 & c_1 \\ a_2x+b_2y+c_2z & b_2 & c_2 \\ a_3x+b_3y+c_3z & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\therefore \quad \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \text{ then } x = \frac{\Delta_1}{\Delta}$$

$$\text{Similarly } \Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ then } y = \frac{\Delta_2}{\Delta}$$

$$\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \text{ then } z = \frac{\Delta_3}{\Delta}$$

$\therefore \dfrac{x}{\Delta_1} = \dfrac{y}{\Delta_2} = \dfrac{z}{\Delta_3} = \dfrac{1}{\Delta}$. This is known as Cramer's Rule.

Matrix Inversion Method

Consider the matrix equation AX = D, where A is non-singular.

Then we can find A^{-1}.

$$AX = D \Leftrightarrow A^{-1}(AX) = A^{-1}D$$

$$(A^{-1}A)X = A^{-1}D$$

$$IX = A^{-1}D$$

$X = A^{-1}D$. From this x, y and z are known.

6. Solve the following simultaneous linear equations by using Cramer's rule.

$3x + 4y + 5z = 18$, $2x - y + 8z = 13$ $5x - 2y + 7z = 20$

Sol. $3x + 4y + 5z = 18$,

$2x - y + 8z = 13$

$5x - 2y + 7z = 20$

$$A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, D = \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$$

Then we can write the given equations in the form of matrix equation as AX = D.

$$\Delta = \det A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix}$$

$= 3\,(-7+16) - 4\,(14-40) + 5\,(-4+5)$

$= 3\,(9) - 4\,(-26) + 5\,(1)$

$= 27 + 104 + 5 = 136 \neq 0$

Hence we can solve the given equation by using Cramer's rule.

$$\Delta_1 = \begin{vmatrix} 18 & 4 & 5 \\ 13 & -1 & 8 \\ 20 & -2 & 7 \end{vmatrix} = 408$$

$$\Delta_2 = \begin{vmatrix} 3 & 18 & 5 \\ 2 & 13 & 8 \\ 5 & 20 & 7 \end{vmatrix} = 136$$

$$\Delta_3 = \begin{vmatrix} 3 & 4 & 18 \\ 2 & -1 & 13 \\ 5 & -2 & 20 \end{vmatrix} = 136$$

Hence by Cramer's rule

$$x = \frac{\Delta_1}{\Delta} = \frac{408}{136} = 3$$

$$y = \frac{\Delta_2}{\Delta} = \frac{136}{136} = 1$$

$$z = \frac{\Delta_3}{\Delta} = \frac{136}{136} = 1$$

The solution of the given system of equations is $\boxed{x = 3,\ y = 1,\ \ z = 1}$

7. Solve the following system of equations by Cramer's rule.

(i) $5x - 6y + 4z = 15,\ \ 7x + 4y - 3z = 19,\ \ 2x + y + 6z = 46$

Sol. (i) $5x - 6y + 4z = 15,$

$7x + 4y - 3z = 19,$

$2x + y + 6z = 46$

$$\det A = \begin{bmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{bmatrix},\ D = \begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\det A = \Delta = \begin{bmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{bmatrix} = 5(24 + 3) + 6\,(42 + 6) + 4\,(7-8)$$

$$= 135 + 288 - 4$$

$$\Delta = 419 \neq 0$$

$$\Delta_1 = \begin{vmatrix} 15 & -6 & 4 \\ 19 & 4 & -3 \\ 46 & 1 & 6 \end{vmatrix} = 1257$$

$$\Delta_2 = \begin{vmatrix} 5 & 15 & 4 \\ 7 & 19 & -3 \\ 2 & 46 & 6 \end{vmatrix} = 1676$$

$$\Delta_3 = \begin{vmatrix} 5 & -6 & 15 \\ 7 & 4 & 19 \\ 2 & 1 & 46 \end{vmatrix} = 2514$$

From Cramer's rule

$$x = \frac{\Delta_1}{\Delta} = \frac{1257}{419} = 3$$

$$y = \frac{\Delta_2}{\Delta} = \frac{1676}{419} = 4$$

$$z = \frac{\Delta_3}{\Delta} = \frac{2514}{419} = 6$$

$$\therefore \boxed{x = 3,\ y = 4,\ z = 6}$$

(ii) $x + y + z = 1$

$2x + 2y + 3z = 6$

$x + 4y + 9z = 3$

Sol. $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$, $D = \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\det A = \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 1(18 - 12) - 1\,(18 - 3) + 1\,(8 - 2)$$

$$\Delta = 6 - 15 + 6 = -3 \neq 0$$

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ 6 & 2 & 3 \\ 3 & 4 & 9 \end{vmatrix} = -21$$

$$\Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 6 & 3 \\ 1 & 3 & 9 \end{vmatrix} = 30$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 6 \\ 1 & 4 & 3 \end{vmatrix} = -12$$

From Cramer's rule

$$x = \frac{\Delta_1}{\Delta} = \frac{-21}{-3} = 7$$

$$y = \frac{\Delta_2}{\Delta} = \frac{30}{-3} = -10$$

$$z = \frac{\Delta_3}{\Delta} = \frac{-12}{-3} = 4$$

$$\therefore \boxed{x = 7,\ y = -10,\ z = 4}$$

(iii) $x - y + 3z = 5$

$4x + 2y - z = 0$

$-x + 3y + z = 5$

Sol. $A = \begin{bmatrix} 1 & -1 & 3 \\ 4 & 2 & -1 \\ -1 & 3 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\det A = \Delta = \begin{vmatrix} 1 & -1 & 3 \\ 4 & 2 & -1 \\ -1 & 3 & 1 \end{vmatrix} = 1(2 + 3) + 1\,(4 - 1) + 3\,(12 + 2)$$

$$= 5 + 3 + 42 = 50 \neq 0$$

$$\Delta_1 = \begin{vmatrix} 5 & -1 & 3 \\ 0 & 2 & -1 \\ 5 & 3 & 1 \end{vmatrix} = 0$$

$$\Delta_2 = \begin{vmatrix} 1 & 5 & 3 \\ 4 & 0 & -1 \\ -1 & 5 & 1 \end{vmatrix} = 50$$

$$\Delta_3 = \begin{vmatrix} 1 & -1 & 5 \\ 4 & 2 & 0 \\ -1 & 3 & 5 \end{vmatrix} = 100$$

From Cramer's rule

$$x = \frac{\Delta_1}{\Delta} = \frac{0}{50} = 0$$

$$y = \frac{\Delta_2}{\Delta} = \frac{50}{50} = 1$$

$$z = \frac{\Delta_3}{\Delta} = \frac{100}{50} = 2$$

$\therefore$ $\boxed{x = 0,\ y = 1,\ z = 2}$

(iv) $x + y + z = 9$

$2x + 5y + 7z = 52$

$2x + y - z = 0$

Sol. $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}$, $D = \begin{bmatrix} 9 \\ 52 \\ 0 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\det A = \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{vmatrix} = 1(-5 - 7) - 1\,(-2 - 14) + 1\,(2 - 10)$$

$$= -12 + 16 - 8 = -4 \neq 0$$

$\Delta = -4$

$$\Delta_1 = \begin{vmatrix} 9 & 1 & 1 \\ 52 & 5 & 7 \\ 0 & 1 & -1 \end{vmatrix} = -4$$

$$\Delta_2 = \begin{vmatrix} 1 & 9 & 1 \\ 2 & 52 & 7 \\ 2 & 0 & -1 \end{vmatrix} = -12$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 9 \\ 2 & 5 & 52 \\ 2 & 1 & 0 \end{vmatrix} = -20$$

From Cramer's rule

$$x = \frac{\Delta_1}{\Delta} = \frac{-4}{-4} = 1$$

$$y = \frac{\Delta_2}{\Delta} = \frac{-12}{-4} = 3$$

$$z = \frac{\Delta_3}{\Delta} = \frac{-20}{-4} = 5$$

$$\boxed{\therefore \quad x = 1, \; y = 3, \;\; z = 5}$$

8. Solve the following systems of equations by using matrix inversion method.

i) $3x + 4y + 5z = 18$, $2x - y - 8z = 13$, $5x - 2y + 7z = 20$

Sol. $3x + 4y + 5z = 18$

$2x - y - 8z = 13$

$5x - 2y + 7z = 20$

Let $A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $D = \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$

Then we can write the given equations in the form

$A\,X = D$

$$\det A = \Delta = \begin{vmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{vmatrix} = 3(-7 + 16) - 4\,(14 - 40) + 5\,(-4 + 5)$$

$$= 27 + 104 + 5 = 136 \neq 0$$

$$\text{Cofactor matrix of A} = \begin{bmatrix} 9 & 26 & 1 \\ -38 & -4 & 26 \\ 37 & -14 & -11 \end{bmatrix}$$

$$\text{Adj A} = \begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}$$

$X = A^{-1} D$

$$X = \left(\frac{\text{Adj A}}{\det A}\right).D$$

$$= \frac{1}{136}\begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}\begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$$

$$= \frac{1}{136}\begin{bmatrix} 408 \\ 136 \\ 136 \end{bmatrix}$$

$$X = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \boxed{x = 3,\ y = 1,\ z = 1}$$

(ii) $2x - y + 3z = 9$

$x + y + z = 6$

$x - y + z = 2$

Sol. $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, D = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$

$$\boxed{AX = D \Rightarrow X = A^{-1}D}$$

$$\det A = \Delta = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 2\,(1 + 1) + 1\,(0 - 0) + 3\,(-1 - 1)$$

$$= 4 - 6 = -2 \neq 0$$

$$\text{Cofactor matrix of A} = \begin{bmatrix} 2 & 0 & -2 \\ -2 & -1 & 1 \\ -4 & 1 & 3 \end{bmatrix}$$

$$\text{Adj A} = \begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix}$$

$$X = A^{-1} D$$

$$X = \left(\frac{\text{Adj A}}{\det A}\right) . D$$

$$X = \frac{1}{-2}\begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$$

$$X = \frac{1}{-2}\begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\boxed{\therefore \quad x = 1, \ y = 2, \ z = 3}$$

(iii) $x + y + z = 1$

$2x + 2y + 3z = 6$

$x + 4y + 9z = 3$

Sol. $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \ D = \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix}$

$$\boxed{AX = D \Rightarrow X = A^{-1}D}$$

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 1\ (18 - 12) - 1\ (18 - 3) + 1\ (8 - 2)$$

$$= 6 - 15 + 6 = -3 \neq 0$$

$\det A \neq 0 = -3$

$$\text{Cofactor matrix of A} = \begin{bmatrix} 6 & -15 & 6 \\ -5 & 8 & -3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\text{Adj A} = \begin{bmatrix} 6 & -5 & 1 \\ -15 & 8 & -1 \\ 6 & -3 & 0 \end{bmatrix}$$

$X = A^{-1} D$

$$X = \left(\frac{\text{Adj A}}{\det A}\right) . D$$

$$X = -\frac{1}{3}\begin{bmatrix} 6 & -5 & 1 \\ -15 & 8 & -1 \\ 6 & -3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} -21 \\ 30 \\ -12 \end{bmatrix} = \begin{bmatrix} 7 \\ -10 \\ 4 \end{bmatrix}$$

$$\boxed{\therefore \quad x = 7,\ y = -10,\ z = 4}$$

(iv) $2x - y + 3z = 8$

$-x + 2y + z = 4$

$3x + y - 4z = 0$

Sol. $A = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \ D = \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}$

$$\boxed{AX = D \Rightarrow X = A^{-1}D}$$

$$\det A = \begin{vmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{vmatrix} = 2(-8-1) + 1(4-3) + 3(-1-6)$$

$$= -18 + 1 - 21 = -38 \neq 0$$

$\det A = -38 \neq 0$

$$\text{Cofactor matrix of A} = \begin{bmatrix} -9 & -1 & -7 \\ -1 & -17 & -5 \\ -7 & -5 & 3 \end{bmatrix}$$

$$\text{Adj A} = \begin{bmatrix} -9 & -1 & -7 \\ -1 & -17 & -5 \\ -7 & -5 & 3 \end{bmatrix}$$

$$X = A^{-1} D$$

$$X = \left(\frac{\text{Adj A}}{\det A}\right) . D$$

$$X = -\frac{1}{38}\begin{bmatrix} -9 & -1 & -7 \\ -1 & -17 & -5 \\ -7 & -5 & 3 \end{bmatrix}\begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}$$

$$X = -\frac{1}{38}\begin{bmatrix} -76 \\ -76 \\ -76 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\boxed{\therefore \quad x = 1,\ y = 1,\ \ z = 1}$$

PRACTISE PROBLEMS

1. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ then show that $A^{-1} = A^3$.

2. Solve the following system of equations by Cramer's rule.

   (i) $2x - y + 3z = 9, x + y + z = 6,\ x - y + z = 2$

   (ii) $2x - y + 3z = 8,\ -x + 2y + z = 4,\ 3x + y - 4z = 0$

   (iii) $2x - y + 8z = 13,\ 3x + 4y + 5z = 18,\ 5x - 2y + 7z = 20$

2. Solve the following system of equations by matrix inversion method.

   (i) $x + y + z = 1, 2x + 2y + 3z = 6,\ x + 4y + 9z = 3$

   (ii) $x - y + 3z = 5,\ 4x + 2y - z = 0,\ -x + 3y + z = 5$

   (iii) $x + y + z = 9,\ 2x + 5y + 7z = 52,\ 2x + y - z = 0$

❖❖❖❖❖

Unit **4**

Addition of Vectors

- **Vector:** A physical quantity which has both magnitude and direction is called a vector. Example: Velocity, displacement, force etc.
- **Scalar:** A physical quantity which has only magnitude is called a scalar. Example: Length, volume, temperature
- **Position Vector:** : Let 'O' and 'P' be any two points in space. Then the vector $\overline{OP}$ having 'O' and 'P' as initial and terminal points respectively, is called the position vector of the point P with respect to 'O'.

  Position vector of P (x,y,z) w.r.t. origin O (0, 0, 0) is denoted by $\bar{r}$.

  Magnitude of $\overline{OP}$ is given by, $\left|\overline{OP}\right| = \left|\bar{r}\right| = \sqrt{x^2 + y^2 + z^2}$

  Note: $\overline{AB} = \overline{OB} - \overline{OA} =$ Position vector of B – Position vector of A.
- **Direction Cosines and Direction Ratios:**

  Let the position vector of point P (x,y,z) w.r.t. origin 'O' be $\overline{OP} = \bar{r}$. Let α, β, γ be the angles made by the vector $\bar{r}$ in the positive direction (counter clockwise direction) of X, Y, Z axes respectively.

  Then $\text{Cos}\alpha, \text{Cos}\beta, \text{Cos}\gamma$ are called the direction cosines of the vector $\bar{r}$.

  These direction cosines are denoted by *l, m, n* respectively.

  i.e. $l = \text{Cos}\alpha$

  $m = \text{Cos}\beta$

  $n = \text{Cos}\gamma$

  Thus the coordinates *x, y, z* of the point P are expressed as (*lr, mr, nr*).

  The numbers *lr, mr, nr* which are proportional to the direction cosines *l, m, n* are called the direction ratios of the vector $\bar{r}$. These direction ratios are denoted by *a, b, c.*

  i.e. $a = lr$

  $b = mr$

  $c = nr$

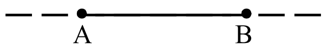
Note: $l^2 + m^2 + n^2 = 1$ but $a^2 + b^2 + c^2 \neq 1$, in general.

- ❖ **Unit Vector:** A vector whose magnitude is unity (i.e. 1 unit) is called a unit vector. It is represented by $\bar{e}$.
- ❖ Unit vector in the direction of a given vector $\bar{a}$ is denoted by $\hat{a}$ and it is given by, $\hat{a} = \frac{\bar{a}}{|\bar{a}|}$
- ❖ The zero vector is denoted by $\bar{0}$ and it is also known as null vector. We can observe that the initial and terminal points coincide for zero vector and its magnitude is the scalar 0.
- ❖ **Like vectors:** If two vectors are having the same direction, then they are called like vectors.
- ❖ **Unlike vectors:** If two vectors are in opposite directions, then they are called unlike vectors.
- ❖ **Negative of a vector:** Let $\bar{a}$ be a vector. The vector having the same magnitude as $\bar{a}$ but having the opposite direction is called the negative vector of $\bar{a}$ and is denoted by $-\bar{a}$.

Note:

1. If $\bar{a} = \overline{AB}$, then $-\bar{a} = \overline{BA}$.
2. Unit vector in the opposite direction of $\bar{a}$ $= \frac{-\bar{a}}{|\bar{a}|}$

- ❖ The line AB is called support of the vector $\overline{AB}$.



- ❖ **Collinear (Parallel) Vectors:** Vectors with same support or parallel supports are called collinear or parallel vectors.

Note: 1. $\bar{a}$, $\bar{b}$ are collinear (parallel) vectors $\Leftrightarrow \bar{a} = \lambda \bar{b}$, where λ is a scalar.

2. The points A, B, C are collinear $\Leftrightarrow \overline{AB} = \lambda \overline{BC}$, where λ is a scalar.

3. If $a_1 i + a_2 j + a_3 k$ and $b_1 i + b_2 j + b_3 k$ are collinear vectors, then $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$.

- ❖ **Coplanar Vectors:** Vectors whose supports are in the same plane or parallel to the same plane are called coplanar vectors.

Note: 1. The points A, B, C, D are coplanar $\Leftrightarrow \overline{AD} = x\overline{AB} + y\overline{AC}$ where *x, y* are scalars.

2. If $\overline{AB} = a_1 i + b_1 j + c_1 k$

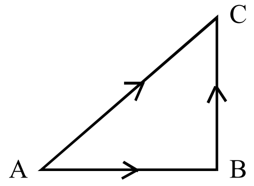
$\overline{AC} = a_2 i + b_2 j + c_2 k$

$\overline{AD} = a_3 i + b_3 j + c_3 k$, then the points A, B, C, D or

$$\overline{AB},\ \overline{AC},\ \overline{AD} \text{ are coplanar} \Leftrightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

- ❖ The vectors which are not coplanar are called non–coplanar vectors.

❖ **Triangle law of vector addition:**



In ΔABC, $\overline{AB}$ and $\overline{BC}$ are two sides,

then their sum is represented by the third side, $\overline{AC}$.

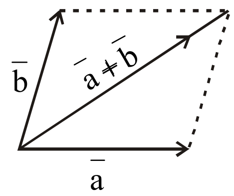
i.e. $\overline{AC} = \overline{AB} + \overline{BC}$

This is known as the triangle law of vector addition.

❖ **Parallelogram law of vector addition:**

If $\overline{a}$, $\overline{b}$ are the adjacent sides of a parallelogram then their sum $\overline{a}+\overline{b}$ is represented by the diagonal of the parallelogram through their common point.

This is known as the parallelogram law of vector addition.



❖ **Properties of vector addition:** For any vectors $\overline{a}$, $\overline{b}$ and $\overline{c}$

(i) $\overline{a}+\overline{b} = \overline{b}+\overline{a}$ (Commutative property)

(ii) $\overline{a}+(\overline{b}+\overline{c}) = (\overline{a}+\overline{b})+\overline{c}$ (Associative property)

(iii) $\overline{a}+\overline{0} = \overline{0}+\overline{a} = \overline{a}$ (Identity property)

Here, the zero vector $\overline{0}$ is called the additive identity for the vector addition.

❖ Let $\overline{a}, \overline{b}$ be two vectors, then

(i) $|\overline{a}+\overline{b}| \le |\overline{a}|+|\overline{b}|$

(ii) $||\overline{a}|-|\overline{b}|| \le |\overline{a}-\overline{b}|$

Note: Equality holds if and only if $\overline{a}$ and $\overline{b}$ are like vectors.

❖ If a point P divides the line segment joining the points A($\overline{a}$) and B($\overline{b}$) in the ratio m : n, then the position vector of P is $\dfrac{m\overline{b} + n\overline{a}}{m+n}$.

❖ **Linear combination of vectors:** Let $\overline{a_1}, \overline{a_2}, \overline{a_3}$ .... $\overline{a_n}$ be vectors and x_1, x_2, x_3 ...., x_n be scalars. Then the vector $x_1\overline{a_1} + x_2\overline{a_2} + x_3\overline{a_3} + ... + x_n\overline{a_n}$ is called a linear combination of the vectors $\overline{a_1}, \overline{a_2}, \overline{a_3}$ ...., $\overline{a_n}$.

❖ Vector equation of the straight line passing through the point $A(\overline{a})$ and parallel to the vector $\overline{b}$ is $\overline{r} = \overline{a} + t\overline{b}$, $t \in R$

❖ Vector equation of the straight line passing through two points $A(\overline{a})$ and $B(\overline{b})$ is, $\overline{r} = (1-t)\overline{a} + t\overline{b}$, $t \in R$

❖ Vector equation of the plane passing through a point $A(\overline{a})$ and parallel to the vectors $\overline{b}, \overline{c}$ is $\overline{r} = \overline{a} + t\overline{b} + s\overline{c}$, $t, s \in R$

❖ Vector equation of the plane passing through three points $A(\bar{a})$, $B(\bar{b})$ and parallel to the vector $\bar{c}$ is $\bar{r} = (1\text{-}t)\,\bar{a} + t\bar{b} + s\bar{c}$, $t, s \in R$

❖ Vector equation of the plane passing through three points $A(\bar{a})$, $B(\bar{b})$ and $C(\bar{c})$ is $\bar{r} = (1\text{-}t\text{-}s)\,\bar{a} + t\bar{b} + s\bar{c}$, $t, s \in R$

VERY SHORT ANSWER TYPE QUESTIONS (2 MARKS)

1) Find the unit vector in the direction of vector $\bar{a} = 2i + 3j + k$.

Sol. $\bar{a} = 2i + 3j + k$

$|\bar{a}| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{4+9+1} = \sqrt{14}$

$\therefore$ Unit vector in the direction of $\bar{a}$ is $\hat{a} = \dfrac{\bar{a}}{|\bar{a}|} = \dfrac{2i+3j+k}{\sqrt{14}}$

$\Rightarrow \hat{a} = \dfrac{2}{\sqrt{14}}i + \dfrac{3}{\sqrt{14}}j + \dfrac{1}{\sqrt{14}}k$

2) Let $\bar{a} = i + 2j + 3k$, $\bar{b} = 3i + j$. Find the unit vector in the direction of $\bar{a} + \bar{b}$.

Sol. $\bar{a} = i + 2j + 3k$

$\bar{b} = 3i + j$

$\bar{a} + \bar{b} = i + 2j + 3k + 3i + j$

$\therefore \bar{a} + \bar{b} = 4i + 3j + 3k$

$|\bar{a} + \bar{b}| = \sqrt{4^2 + 3^2 + 3^2} = \sqrt{16+9+9} = \sqrt{34}$

$\therefore$ Unit vector in the direction of $|\bar{a} + \bar{b}|$ = $\dfrac{\bar{a} + \bar{b}}{|\bar{a} + \bar{b}|} = \dfrac{4i+3j+3k}{\sqrt{34}}$

$= \dfrac{1}{\sqrt{34}}(4i+3j+3k)$

3) Find the unit vector in the direction of the sum of the vectors $\bar{a} = 2i + 2j - 5k$ and $\bar{b} = 2i + j + 3k$.

Sol. $\bar{a} = 2i + 2j - 5k$, $\bar{b} = 2i + j + 3k$

$\bar{a} + \bar{b} = 2i + 2j - 5k + 2i + j + 3k$

$\bar{a} + \bar{b} = 4i + 3j - 2k$

$|\bar{a} + \bar{b}| = \sqrt{4^2 + 3^2 + (-2)^2} = \sqrt{16+9+4} = \sqrt{29}$

$\therefore$ Unit vector in the direction of sum of $\bar{a}$ and $\bar{b} = \dfrac{\bar{a}+\bar{b}}{|\bar{a}+\bar{b}|} = \dfrac{4i+3j-2k}{\sqrt{29}}$

4) Let $\bar{a} = 2i+4j-5k$, $\bar{b} = i+j+k$ and $\bar{c} = j+2k$. Find the unit vector in the opposite direction of $\bar{a}+\bar{b}+\bar{c}$.

Sol. $\bar{a} = 2i+4j-5k$

$\bar{b} = i+j+k$

$\bar{c} = j+2k$

$\bar{a}+\bar{b}+\bar{c} = (2i+4j-5k)+(i+j+k)+(j+2k)$

$\bar{a}+\bar{b}+\bar{c} = 3i+6j-2k$

$|\bar{a}+\bar{b}+\bar{c}| = \sqrt{3^2+6^2+(-2)^2} = \sqrt{9+36+4} = \sqrt{49} = 7$

$\therefore$ Unit vector in the opposite direction of $\bar{a}+\bar{b}+\bar{c}$

$= -\dfrac{(\bar{a}+\bar{b}+\bar{c})}{|\bar{a}+\bar{b}+\bar{c}|}$

$= -\dfrac{(3i+6j-2k)}{7}$

5) If the position vectors of the points A, B and C are $-2i+j-k$, $-4i+2j+2k$ and $6i-3j-13k$ respectively and $\overline{AB} = \lambda\overline{AC}$, then find the value of λ.

Sol. Let 'O' be the origin.

Then, $\overline{OA} = -2i+j-k$

$\overline{OB} = -4i+2j+2k$

$\overline{OC} = 6i-3j-13k$

$\therefore \overline{AB} = \overline{OB} - \overline{OA} = (-4i+2j+2k)-(-2i+j-k)$

$= -4i+2j+2k+2i-j+k$

$\therefore \overline{AB} = -2i+j+3k$

$\therefore \overline{AC} = \overline{OC} - \overline{OA} = (6i-3j-13k)-(-2i+j-k)$

$= 6i-3j-13k+2i-j+k$

$= 8i-4j-12k$

$\overline{AC} = -4(-2i+j+3k)$

$\overline{AC} = -4.\ \overline{AB}$ $\quad \left[\because \overline{AB} = -2i+j+3k\right]$

$\Rightarrow -4\overline{AB} = \overline{AC}$

$$\overline{AB} = -\frac{1}{4}\overline{AC}$$

Comparing with, $\overline{AB} = \lambda\ \overline{AC}$ we get,

$$\lambda = -\frac{1}{4}$$

6) If $\overline{OA} = i + j + k$, $\overline{AB} = 3i - 2j + k$, $\overline{BC} = i + 2j - 2k$ and $\overline{CD} = 2i + j + 3k$, then find the vector $\overline{OD}$.

Sol. $\overline{OA} = i + j + k$

$\overline{AB} = 3i - 2j + k$

$\overline{BC} = i + 2j - 2k$

$\overline{CD} = 2i + j + 3k$

$\because\ \overline{OA} + \overline{AB} + \overline{BC} + \overline{CD} = \overline{OD}$

$\Rightarrow \overline{OD} = \overline{OA} + \overline{AB} + \overline{BC} + \overline{CD}$

$= (i + j + k) + (3i - 2j + k) + (i + 2j - 2k) + (2i + j + 3k)$

$\therefore\ \overline{OD} = 7i + 2j + 3k$

7) Write direction ratios of the vector $\bar{a} = i + j - 2k$ and hence calculate its direction cosines.

Sol. Let $\bar{r} = \bar{a} = i + j - 2k$

Let $\bar{a}, \bar{b}, \bar{c}$ be the direction ratios of vector $\bar{r} = xi + yj + zk$

Then the values of a, b, c are just the respective components x, y and z of the vector.

Hence, a = 1, b = 1, c = –2

If *l, m, n* are the direction cosines of the given vector, then

$$|\bar{r}| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{1+1+4} = \sqrt{6}$$

$$l = \frac{a}{|\bar{r}|} = \frac{1}{\sqrt{6}}$$

$$m = \frac{b}{|\bar{r}|} = \frac{1}{\sqrt{6}}$$

$$n = \frac{c}{|\bar{r}|} = \frac{-2}{\sqrt{6}}$$

$\therefore$ The direction cosines are $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$

8) If the vectors $-3i + 4j + \lambda k$ and $\mu i + 8j + 6k$ are collinear vectors, then find λ and μ.

Sol. The vectors, $-3i + 4j + \lambda k$ and $\mu i + 8j + 6k$ are collinear.

$\Rightarrow \quad \frac{-3}{\mu} = \frac{4}{8} = \frac{\lambda}{6}$

$\Rightarrow \quad \frac{-3}{\mu} = \frac{1}{2} = \frac{\lambda}{6}$

$\Rightarrow \quad \frac{-3}{\mu} = \frac{1}{2}$ and $\frac{1}{2} = \frac{\lambda}{6}$

$\Rightarrow \quad \mu = 2(-3) \qquad 2\lambda = 6(1)$

$\Rightarrow \quad \mu = -6 \qquad \lambda = \frac{6}{2} = 3$

$\therefore \ \lambda = 3$ and $\mu = -6$

9) Find the vector equation of the line passing through the point **2i + 3j + k** and parallel to the vector **4i – 2j + 3k.**

Sol. Let $\bar{a} = 2i + 3j + k$

$\bar{b} = 4i - 2j + 3k$

Vector equation of the line passing through $\bar{a}$ and parallel to $\bar{b}$ is,

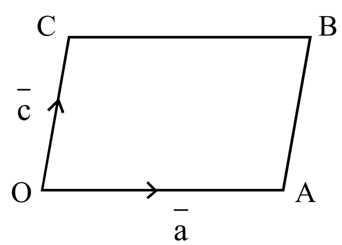
$\bar{r} = \bar{a} + t\bar{b}, \ t \in R$

$\bar{r} = (2i + 3j + k) + t(4i - 2j + 3k)$

$\Rightarrow \quad \bar{r} = (2 + 4t)i + (3-2t)j + (1+3t)k$

10) OABC is a parallelogram. If $\overline{OA} = \bar{a}$ and $\overline{OC} = \bar{c}$, find the vector equation of the side $\overline{BC}$.

Sol. OABC is a parallelogram in which,



$\overline{OA} = \bar{a}$

$\overline{OC} = \bar{c} \ \Rightarrow \ \overline{AB} = \bar{c}$

$\Rightarrow \ \overline{OB} - \overline{OA} = \bar{c}$

$\Rightarrow \ \overline{OB} = \bar{c} + \overline{OA}$

$\Rightarrow \ \overline{OB} = \bar{c} + \bar{a}$

$\therefore \ \overline{OB} = \bar{a} + \bar{c}$

$\therefore$ The vector equation of $\overline{BC}$, $\bar{r} = (1-t)\bar{c} + t(\bar{a} + \bar{c}), \ t \in R$

$\Rightarrow \quad \bar{r} = (1-t+t)\bar{c} + t\,\bar{a}$

$\Rightarrow \quad \bar{r} = \bar{c} + t\,\bar{a}$

11) Find the vector equation of the line joining the points 2i + j + 3k and –4i + 3j – k.

Sol. Let $\bar{a}$ = 2i + j + 3k

$\bar{b}$ = –4i + 3j – k

Vector equation of line passing through $\bar{a}$ and $\bar{b}$ is

$\bar{r} = (1-t)\bar{a} + t\bar{b}$, $t \in R$

$\Rightarrow$ $\bar{r}$ = (1 – t) (2i + j + 3k) + t(–4i + 3j – k)

$\Rightarrow$ $\bar{r}$ = (2 – 2t – 4t)i + (1 – t + 3t)j + (3 – 3t – t)k

$\Rightarrow$ $\bar{r}$ = (2 – 6t)i + (1 + 2t)j + (3 – 4t)k

$\Rightarrow$ $\bar{r}$ = 2(1 – 3t)i + (1 + 2t)j + (3 – 4t)k

12) Find the vector equation of the plane passing through the points i – 2j + 5k, –5j – k and –3i + 5j.

Sol. Let $\bar{a}$ = i – 2j + 5k

$\bar{b}$ = –5j – k

$\bar{c}$ = –3i + 5j

$\therefore$ Vector equation of the plane passing through $\bar{a}$, $\bar{b}$ and $\bar{c}$ is,

$\bar{r} = (1\text{-}t\text{-}s)\bar{a} + t\bar{b} + s\bar{c}$, $t, s \in R$

$\Rightarrow$ $\bar{r}$ = (1 – t – s) (i – 2j + 5k) + t(–5j – k) + s(–3i + 5j)

13) Find the vector equation of the plane passing through the points (0, 0, 0), (0, 5, 0) and (2, 0, 1).

Sol. $\bar{a}$ = 0.i + 0.j + 0.k = $\bar{0}$

$\bar{b}$ = 0.i + 5j + 0.k = 5j

$\bar{c}$ = 2.i + 0.j + 1.k = 2i + k

Vector equation of the plane passing through $\bar{a}$, $\bar{b}$ and $\bar{c}$ is,

$\bar{r} = (1\text{-}t\text{-}s)\bar{a} + t\bar{b} + s\bar{c}$, $t, s \in R$

$\Rightarrow$ $\bar{r}$ = (1 – t – s) $\bar{0}$ + t(5j) + s(2i + k)

$\Rightarrow$ $\bar{r}$ = (5t)j + s(2i + k)

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

1) Show that the points A(2i – j + k), B(i – 3j – 5k), C(3i – 4j – 4k) are the vertices of a right angle triangle.

Sol. Let 'O' be the origin, then

$\overline{OA}$ = 2i – j + k

$\overline{OB} = i - 3j - 5k$

$\overline{OC} = 3i - 4j - 4k$

$\therefore \overline{AB} = \overline{OB} - \overline{OA} = (i - 3j - 5k) - (2i - j + k)$

$= (1-2)i + (-3+1)j + (-5-1)k$

$\overline{AB} = -i - 2j - 6k$

$\Rightarrow \quad |\overline{AB}| = \sqrt{(-1)^2 + (-2)^2 + (-6)^2} = \sqrt{1+4+36} = \sqrt{41}$

$\overline{BC} = \overline{OC} - \overline{OB} = (3i - 4j - 4k) - (i - 3j - 5k)$

$\overline{BC} = (3 - 1)i + (-4 + 3)j + (-4 + 5)k = 2i - j + k$

$|\overline{BC}| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{4+1+1} = \sqrt{6}$

$\overline{CA} = \overline{OA} - \overline{OC} = (2i - j + k) - (3i - 4j - 4k)$

$\overline{CA} = (2 - 3)i + (-1 + 4)j + (1 + 4)k = -i + 3j + 5k$

$|\overline{CA}| = \sqrt{(-1)^2 + 3^2 + 5^2} = \sqrt{1+9+25} = \sqrt{35}$

$|\overline{AB}|^2 = (\sqrt{41})^2 = (\sqrt{6})^2 + (\sqrt{35})^2$

$\Rightarrow \quad |\overline{AB}|^2 = |\overline{BC}|^2 + |\overline{CA}|^2$

$\Rightarrow$ A, B, C are the vertices of a right angle triangle.

2) Is the triangle formed by the vectors $3i + 5j + 2k$, $2i - 3j - 5k$ and $-5i - 2j + 3k$ equilateral?

Sol. In ΔABC, let $\overline{AB} = 3i + 5j + 2k$

$\overline{BC} = 2i - 3j - 5k$

$\overline{CA} = -5i - 2j + 3k$

$|\overline{AB}| = \sqrt{3^2 + 5^2 + 2^2} = \sqrt{9+25+4} = \sqrt{38}$

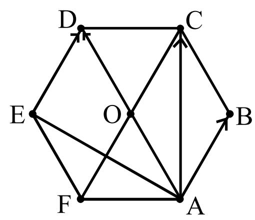
$|\overline{BC}| = \sqrt{2^2 + (-3)^2 + (-5)^2} = \sqrt{4+9+25} = \sqrt{38}$

$|\overline{CA}| = \sqrt{(-5)^2 + (-2)^2 + 3^2} = \sqrt{25+4+9} = \sqrt{38}$

$\therefore \quad |\overline{AB}| = |\overline{BC}| = |\overline{CA}| \Rightarrow \Delta ABC$ is an equilateral triangle.

3) If centre of the regular hexagon ABCDEF is 'O, then show that

AB + AC + AD + AE = 3 AD = 6 AO.



Sol. From figure, AB + AC + AD + AE + AF

= (AB + AE) + AD + (AC + AF)

$= (AE + ED) + AD + (AC + CD)$

($\because$ AB = ED, AF = CD) (from figure)

$= AD + AD + AD = 3\,AD$

$= 6\,AO$ ($\because$ O centre, OD = AO)

4) $\bar{a}, \bar{b}, \bar{c}$ are non–coplanar vectors. Prove that the following four points are coplanar.

(i) $-\bar{a} + 4\bar{b} - 3\bar{c}$, $3\bar{a} + 2\bar{b} - 5\bar{c}$, $-3\bar{a} + 8\bar{b} - 5\bar{c}$, $-3\bar{a} + 2\bar{b} + \bar{c}$

(ii) $6\bar{a} + 2\bar{b} - \bar{c}$, $2\bar{a} - \bar{b} + 3\bar{c}$, $-\bar{a} + 2\bar{b} - 4\bar{c}$, $-12\bar{a} - \bar{b} - 3\bar{c}$

Sol. (i) Let 'O' be the origin. Then the position vectors of A, B, C, D are

$\overline{OA} = -\bar{a} + 4\bar{b} - 3\bar{c}$

$\overline{OB} = 3\bar{a} + 2\bar{b} - 5\bar{c}$

$\overline{OC} = -3\bar{a} + 8\bar{b} - 5\bar{c}$

$\overline{OD} = -3\bar{a} + 2\bar{b} + \bar{c}$

$\overline{AB} = \overline{OB} - \overline{OA} = (3\bar{a} + 2\bar{b} - 5\bar{c}) - (-\bar{a} + 4\bar{b} - 3\bar{c}) = 4\bar{a} - 2\bar{b} - 2\bar{c}$

$\overline{AC} = \overline{OC} - \overline{OA} = (-3\bar{a} + 8\bar{b} - 5\bar{c}) - (-\bar{a} + 4\bar{b} - 3\bar{c}) = -2\bar{a} + 4\bar{b} - 2\bar{c}$

$\overline{AD} = \overline{OD} - \overline{OA} = (-3\bar{a} + 2\bar{b} + \bar{c}) - (-\bar{a} + 4\bar{b} - 3\bar{c}) = -2\bar{a} - 2\bar{b} + 4\bar{c}$

A, B, C, D are coplanar $\Leftrightarrow \begin{vmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{vmatrix} = 0$

$$\begin{vmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{vmatrix} = 4(16-4) + 2(-8-4) - 2(4+8)$$

$= 4(12) + 2(-12) - 2(12)$

$= 48 - 24 - 24$

$= 0$

$\Rightarrow$ A, B, C, D are coplanar.

Second Method:

A, B, C, D are coplanar $\Leftrightarrow \overline{AB}, \overline{AC}, \overline{AD}$ are coplanar.

$\Leftrightarrow \overline{AB} = x\overline{AC} + y\overline{AD}$

where x, y are scalars.

$\Rightarrow 4\bar{a} - 2\bar{b} - 2\bar{c} = x(-2\bar{a} + 4\bar{b} - 2\bar{c}) + y(-2\bar{a} - 2\bar{b} + 4\bar{c})$

$\Rightarrow 4\bar{a} - 2\bar{b} - 2\bar{c} + 2\bar{a}x - 4\bar{b}x + 2\bar{c}x + 2\bar{a}y + 2\bar{b}y - 4\bar{c}y = 0$

$\Rightarrow (4+2x+2y)\bar{a} + (-2-4x+2y)\bar{b} + (-2+2x-4y)\bar{c} = 0$

$\because \bar{a}, \bar{b}, \bar{c}$ are non–coplanar

$\Rightarrow \quad 4 + 2x + 2y = 0 \quad$ ....(1)

$-2 - 4x + 2y = 0 \quad$ ....(2)

$-2 + 2x - 4y = 0 \quad$ ....(3)

Solving (1) and (2)

$$\begin{array}{l} 2x + 2y + 4 = 0 \\ -4x + 2y - 2 = 0 \\ + \quad - \quad\quad + \\ \hline 6x \quad\quad + 6 = 0 \end{array}$$

$x = -6/6 = -1$

Substituting $x = -1$ in equation (1), we get

$4 + 2(-1) + 2y = 0$

$4 - 2 + 2y = 0$

$2 + 2y = 0$

$2y = -2$

$y = -2 / 2 = -1$

Substituting $x = -1, y = -1$ in equation (3), we get

$-2 + 2(-1) - 4(-1) = -2 - 2 + 4 = -4 + 4 = 0$

$\therefore \overline{AB}, \overline{AC}, \overline{AD}$ are coplanar.

$\Rightarrow$ A, B, C, D are coplanar.

$\therefore$ Given points are coplanar.

(ii) Let 'O' be the origin. Then the position vectors of A, B, C, D are

$\overline{OA} = 6\bar{a} + 2\bar{b} - \bar{c}$

$\overline{OB} = 2\bar{a} - \bar{b} + 3\bar{c}$

$\overline{OC} = -\bar{a} + 2\bar{b} - 4\bar{c}$

$\overline{OD} = -12\bar{a} - \bar{b} - 3\bar{c}$ respectively

$\overline{AB} = \overline{OB} - \overline{OA} = (2\bar{a} - \bar{b} + 3\bar{c}) - (6\bar{a} + 2\bar{b} - \bar{c}) = -4\bar{a} - 3\bar{b} + 4\bar{c}$

$\overline{AC} = \overline{OC} - \overline{OA} = (-\bar{a} + 2\bar{b} - 4\bar{c}) - (6\bar{a} + 2\bar{b} - \bar{c}) = -7\bar{a} - 3\bar{c}$

$\overline{AD} = \overline{OD} - \overline{OA} = (-12\bar{a} - \bar{b} - 3\bar{c}) - (6\bar{a} + 2\bar{b} - \bar{c}) = -18\bar{a} - 3\bar{b} - 2\bar{c}$

A, B, C, D are coplanar $\Leftrightarrow \begin{vmatrix} -4 & -3 & 4 \\ -7 & 0 & -3 \\ -18 & -3 & -2 \end{vmatrix} = 0$

$$\begin{vmatrix} -4 & -3 & 4 \\ -7 & 0 & -3 \\ -18 & -3 & -2 \end{vmatrix} = -4(0-9) + 3(14-54) + 4(21-0)$$

$= 36 + 3(-40) + 4(21)$

$= 36 - 120 + 84$

$= 120 - 120 = 0$

$\Rightarrow$ A, B, C, D are coplanar.

5) If i, j, k are unit vectors along the positive direction of the coordinate axes, then show that the four points 4i + 5j + k, –j–k, 3i +9 j + 4k and –4i + 4j + 4k are coplanar.

Sol. Let 'O' be the origin and A, B, C, D be the given points.

Then, $\overline{OA} = 4i + 5j + k$

$\overline{OB} = -j - k$

$\overline{OC} = 3i + 9j + 4k$

$\overline{OD} = -4i + 4j + 4k$

$\overline{AB} = \overline{OB} - \overline{OA} = (-j - k) - (4i + 5j + k) = -4i - 6j - 2k$

$\overline{AC} = \overline{OC} - \overline{OA} = (3i + 9j + 4k) - (4i + 5j + k) = -i + 4j + 3k$

$\overline{AD} = \overline{OD} - \overline{OA} = (-4i + 4j + 4k) - (4i + 5j + k) = -8i - j + 3k$

A, B, C, D are coplanar $\Leftrightarrow \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix} = 0$

$\because \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix} = -4(12+3) + 6(-3+24) - 2(1+32)$

$= -4(15) + 6(21) - 2(33)$

$= -60 + 126 - 66$

$= -126 + 126$

$= 0$

$\Rightarrow$ A, B, C, D are coplanar.

6) If $\bar{a}$, $\bar{b}$, $\bar{c}$ are non–coplanar vectors, then test for the collinearity of the following points whose position vectors are given by

(i) $\bar{a} - 2\bar{b} + 3\bar{c}$, $2\bar{a} + 3\bar{b} - 4\bar{c}$, $-7\bar{b} + 10\bar{c}$

(ii) $3\bar{a} - 4\bar{b} + 3\bar{c}$, $-4\bar{a} + 5\bar{b} - 6\bar{c}$, $4\bar{a} - 7\bar{b} + 6\bar{c}$

Sol. (i) Let 'O' be the origin and A, B, C be the given points.

$\overline{OA} = \bar{a} - 2\bar{b} + 3\bar{c}$

$\overline{OB} = 2\bar{a} + 3\bar{b} - 4\bar{c}$

$\overline{OC} = -7\bar{b} + 10\bar{c}$

$\overline{AB} = \overline{OB} - \overline{OA} = (2\bar{a} + 3\bar{b} - 4\bar{c}) - (\bar{a} - 2\bar{b} + 3\bar{c}) = \bar{a} + 5\bar{b} - 7\bar{c}$ .....(1)

$\overline{BC} = \overline{OC} - \overline{OB} = (-7\bar{b} + 10\bar{c}) - (2\bar{a} + 3\bar{b} - 4\bar{c}) = -2\bar{a} - 10\bar{b} + 14\bar{c}$

$\overline{BC} = -2(\bar{a} + 5\bar{b} - 7\bar{c})$

$\overline{BC} = -2\,\overline{AB}$ [$\because$ from (1)]

$\Rightarrow$ $\overline{BC} = 2\,\overline{BA}$

$\Rightarrow$ A, B, C are collinear.

(ii) Let 'O' be the origin A, B, C be the given points

$\overline{OA} = 3\bar{a} - 4\bar{b} + 3\bar{c}$

$\overline{OB} = -4\bar{a} + 5\bar{b} - 6\bar{c}$

$\overline{OC} = 4\bar{a} - 7\bar{b} + 6\bar{c}$

$\overline{AB} = \overline{OB} - \overline{OA} = (-4\bar{a} + 5\bar{b} - 6\bar{c}) - (3\bar{a} - 4\bar{b} + 3\bar{c}) = -7\bar{a} + 9\bar{b} - 9\bar{c}$

$\overline{BC} = \overline{OC} - \overline{OB} = (4\bar{a} - 7\bar{b} + 6\bar{c}) - (-4\bar{a} + 5\bar{b} - 6\bar{c}) = 8\bar{a} - 12\bar{b} + 12\bar{c}$

$\overline{AB} \neq \lambda\,\overline{BC}$, where λ is a scalar.

$\Rightarrow$ A,B, C are non–collinear.

7) If the points whose position vectors are $3i - 2j - k$, $2i + 3j - 4k$, $-i + j + 2k$ and $4i + 5j + \lambda k$ are coplanar, then show that $\lambda = \frac{-146}{17}$.

Sol. Let 'O' be the origin and A, B, C, D be given points.

$\overline{OA} = 3i - 2j - k$

$\overline{OB} = 2i + 3j - 4k$

$\overline{OC} = -i + j + 2k$

$\overline{OD} = 4i + 5j + \lambda k$ respectively.

$\Rightarrow$ $\overline{AB} = \overline{OB} - \overline{OA} = (2i + 3j - 4k) - (3i - 2j - k) = -i + 5j - 3k$

$\overline{AC} = \overline{OC} - \overline{OA} = (-i + j + 2k) - (3i - 2j - k) = -4i + 3j + 3k$

$\overline{AD} = \overline{OD} - \overline{OA} = (4i + 5j + \lambda k) - (3i - 2j - k) = i + 7j + (\lambda + 1)k$

A, B, C, D are coplanar $\Leftrightarrow \begin{vmatrix} -1 & 5 & -3 \\ -4 & 3 & 3 \\ 1 & 7 & \lambda+1 \end{vmatrix} = 0$

$-1[3(\lambda + 1) - 21] - 5[-4(\lambda + 1) - 3] - 3[-28-3] = 0$

$-1(3\lambda + 3 - 21) - 5(-4\lambda - 4 - 3) - 3(-31) = 0$

$-1(3\lambda-18)-5(-4\lambda-7)+93=0$

$-3\lambda+18+20\lambda+35+93=0$

$17\lambda+146=0$

$17\lambda=-146$

$\therefore\ \lambda=-\frac{146}{17}$

8) Find the vector equation of the plane which passes through the points 2i + 4j + 2k, 2i + 3j + 5k and parallel to the vector 3i – 2j + k. Also find the point where this plane meets the line joining the points 2i + j + 3k and 4i – 2j + 3k.

Sol. Let $\bar{a}$ = 2i + 4j + 2k

$\bar{b}$ = 2i + 3j + 5k

$\bar{c}$ = 3i – 2j + k

$\therefore$ Vector equation of plane passing through $\bar{a}$, $\bar{b}$ and parallel to $\bar{c}$ is given by,

$\bar{r}=(1-t)\bar{a}+t\bar{b}+s\bar{c},\quad t, s \in R$

$\bar{r}$ = (1– t) (2i + 4j + 2k) + t(2i + 3j + 5k) + s(3i – 2j + k)

$\bar{r}$ = (2 – 2t + 2t + 3s)i + (4 – 4t + 3t – 2s)j + (2 – 2t + 5t + s) k

$\bar{r}$ = (2 + 3s)i + (4 – t – 2s) j + (2 + 3t + s) k .... (1)

Let $\bar{p}$ = 2i + j + 3k

$\bar{q}$ = 4i – 2j + 3k

Vector equation of line passing through $\bar{p}$ and $\bar{q}$ is given by,

$\bar{r}=(1-x)\bar{p}+x\,\bar{q},\quad x \in R$

$\bar{r}$ = (1 – x) (2i + j + 3k) + x(4i – 2j + 3k)

$\bar{r}$ = (2 – 2x + 4x)i + (1 – x – 2x) j + (3 – 3x + 3x)k

$\bar{r}$ = (2 + 2x)i + (1 – 3x) j + 3k ....(2)

Equating the corresponding coefficients of i, j, k from (1) & (2), we get

2 + 3s = 2 + 2x $\Rightarrow$ 2x – 3s = 0. ....(3)

4 – t – 2s = 1 – 3x $\Rightarrow$ 3x – 2s – t = –3 ....(4)

2 + 3t + s = 3 $\Rightarrow$ s + 3t = 1

$\Rightarrow 3t = 1 - s \Rightarrow t = \frac{1-s}{3}$

Substituting 't' value in equation (4), we get

$3x-2s-\left(\frac{1-s}{3}\right)=-3$

9x – 6s – 1 + s = –9

$\Rightarrow$ 9x – 5s = –8 ....(5)

Solving (3) & (5), we get

$(2x - 3s = 0) \times 5 \quad \Rightarrow \quad 10x - 15s = 0$

$(9x - 5s = -8) \times -3 \quad \Rightarrow \quad -27x + 15s = 24$

$\qquad\qquad\qquad\qquad\qquad -17x \quad = 24$

$x = \frac{-24}{17}$

Substituting $x = \frac{-24}{17}$ in (2), we get

$\bar{r} = \left(2 + 2\left(\frac{-24}{17}\right)\right)i + \left(1 - 3\left(\frac{-24}{17}\right)\right)j + 3k$

$\bar{r} = \left(2 - \frac{48}{17}\right)i + \left(1 + \frac{72}{17}\right)j + 3k$

$\bar{r} = \left(\frac{34 - 48}{17}\right)i + \left(\frac{17 + 72}{17}\right)j + 3k$

$\Rightarrow \quad \bar{r} = \frac{-14}{17}i + \frac{89}{17}j + 3k$

$\therefore$ Point of intersection of plane and line = $\left(\frac{-14}{17}, \frac{89}{17}, 3\right)$

9) Find the vector equation of the plane passing through points 4i – 3j – k, 3i + 7j – 10k and 2i + 5j – 7k and show that the point i + 2j – 3k lies in the plane.

Sol. Let $\bar{a}$ = 4i – 3j – k

$\bar{b}$ = 3i + 7j – 10k

$\bar{c}$ = 2i + 5j – 7k

$\bar{d}$ = i + 2j – 3k

Vector equation of plane passing through $\bar{a}$, $\bar{b}$ and $\bar{c}$ is

$\bar{r} = (1 - t - s)\bar{a} + t\bar{b} + s\bar{c} \qquad t, s \in R$

$\bar{r}$ = (1 – t – s) (4i – 3j – k) + t (3i + 7j – 10k) + s(2i + 5j – 7k)

If the point $\bar{d}$ lies on this plane, then

i + 2j – 3k = (1 – t – s) (4i – 3j –k) + t(3i + 7j – 10k) + s(2i + 5j – 7k)

i + 2j – 3k = (4 – 4t – 4s + 3t + 2s)i + (–3 + 3t + 3s + 7t + 5s)j + (–1 + t + s – 10t – 7s)k

i + 2j – 3k = (4 – t – 2s)i + (–3 + 10t + 8s)j + (–1 – 9t – 6s)k

Equating the coefficient of i, j, k on both sides, we get

$4 - t - 2s = 1 \quad \Rightarrow \quad t + 2s = 3 \quad$ ....(1)

$-3 + 10t + 8s = 2 \Rightarrow \quad 10t + 8s = 5 \quad$ ....(2)

$-1 - 9t - 6s = -3 \quad \Rightarrow \quad 9t + 6s = 2$ ....(3)

Solving (1) & (2)

$(t + 2s = 3) \times -4 \Rightarrow \quad -4t - 8s = -12$

$10t + 8s = 5 \quad \Rightarrow \quad 10t + 8s = 5$

$6t \quad = -7 \Rightarrow t = \frac{-7}{6}$

From (1) $\quad t + 2s = 3$

$\frac{-7}{6} + 2s = 3$

$2s = 3 + \frac{7}{6} = \frac{18+7}{6}$

$2s = \frac{25}{6} \Rightarrow s = \frac{25}{12}$

From (3)

LHS = 9t + 6s

$= 9\left(\frac{-7}{6}\right) + 6\left(\frac{25}{12}\right) = \frac{-21}{2} + \frac{25}{2} = \frac{-21+25}{2} = \frac{4}{2} = 2$ = R.H.S.

$\therefore\ t = \frac{-7}{6}, s = \frac{25}{12}$ satisfy (1), (2) and (3) equations.

$\Rightarrow \bar{d}$ lies on the plane passing through $\bar{a}$, $\bar{b}$ and $\bar{c}$.

10) Show that the line joining the pair of points $6\bar{a} - 4\bar{b} + 4\bar{c}$, $-4\bar{c}$ and the line joining the pair of points $-\bar{a} - 2\bar{b} - 3\bar{c}$, $\bar{a} + 2\bar{b} - 5\bar{c}$ intersect at the point –4c when $\bar{a}$, $\bar{b}$, $\bar{c}$ are non–coplanar vectors.

Sol. Equation of the line joining the first pair of points is,

$\bar{r} = (1 - t)(-4\bar{c}) + t(6\bar{a} - 4\bar{b} + 4\bar{c}), \quad t \in R$

$\bar{r} = (6t)\bar{a} + (-4t)\bar{b} + (-4 + 4t + 4t)\bar{c}$

$\bar{r} = (6t)\bar{a} + (-4t)\bar{b} + (8t - 4)\bar{c}$ ....(1)

Equation of the line joining the second pair of points is,

$\bar{r} = (1 - s)(-\bar{a} - 2\bar{b} - 3\bar{c}) + s(\bar{a} + 2\bar{b} - 5\bar{c}), \quad s \in R$

$\bar{r} = (-1 + s + s)\bar{a} + (-2 + 2s + 2s)\bar{b} + (-3 + 3s - 5s)\bar{c}$

$\bar{r} = (2s - 1)\bar{a} - (4s - 2)\bar{b} + (-2s - 3)\bar{c}$ ....(2)

Equating the corresponding coefficients of $\bar{a}$, $\bar{b}$ and $\bar{c}$ in (1) & (2), we have

$6t = 2s - 1 \quad \Rightarrow \quad 6t - 2s = -1$ ....(3)

$-4t = 4s - 2 \quad \Rightarrow \quad 4t + 4s = 2 \Rightarrow 2t + 2s = 1$ ....(4)

$8t - 4 = -2s - 3 \Rightarrow \quad 8t + 2s = 1$ ....(5)

Solving (3) & (4), we get

$6t - 2s = -1$

$2t + 2s = 1$

$8t \quad = 0 \qquad \Rightarrow t = 0$

From (4) $\qquad 2t + 2s = 1$

$2(0) + 2s = 1$

$2s = 1 \Rightarrow s = \frac{1}{2}$

$t = 0, s = \frac{1}{2}$ satisfy equation (5).

$\therefore$ Substituting the value of $t = 0$ in (1) or $s = \frac{1}{2}$ in (2), the point of intersection of the lines is –4c.

10) Find the point of intersection of the line $\bar{r} = 2\bar{a} + \bar{b} + t(\bar{b} - \bar{c})$ and the plane $\bar{r} = \bar{a} + x(\bar{b} + \bar{c}) + y(\bar{a} + 2\bar{b} - \bar{c})$ where $\bar{a}, \bar{b}, \bar{c}$ are non–coplanar vectors.

Sol. Given line is, $\bar{r} = 2\bar{a} + \bar{b} + t(\bar{b} - \bar{c})$ ....(1)

plane is, $\bar{r} = \bar{a} + x(\bar{b} + \bar{c}) + y(\bar{a} + 2\bar{b} - \bar{c})$ ....(2)

At the point of intersection of the line and the plane, we have,

$2\bar{a} + \bar{b} + t(\bar{b} - \bar{c}) = \bar{a} + x(\bar{b} + \bar{c}) + y(\bar{a} + 2\bar{b} - \bar{c})$

$2\bar{a} + (1+t)\bar{b} - t\bar{c} = (1+y)\bar{a} + (x+2y)\bar{b} + (x-y)\bar{c}$

$\therefore$ On comparing the corresponding coefficients,

$2 = 1 + y \Rightarrow \quad y = 2 - 1 = 1 \Rightarrow y = 1$

$1 + t = x + 2y \Rightarrow 1 + t = x + 2(1) \Rightarrow t - x = 1$ ....(3)

$-t = x - y \Rightarrow -t = x - 1 \Rightarrow t + x = 1$ .... (4)

Solving (3) & (4)

$t - x = 1$

$t + x = 1$

$2t \quad = 2$

$t = 1$

From (4) $\qquad t + x = 1$

$1 + x = 1$

$\Rightarrow x = 1 - 1$

$\Rightarrow x = 0$

Substituting t = 1 in (1) or substituting x = 0, y = 1 in (2), we get the point of intersection of (1) & (2) as $2\bar{a} + 2\bar{b} - \bar{c}$.

❖❖❖❖❖

Unit **5**

Product of Vectors

Scalar or Dot Product of two vectos

Let $\bar{a}$ and $\bar{b}$ be two vectos. The scalar (or dot) product of $\bar{a}$ and $\bar{b}$, written as , $\bar{a}\,.\,\bar{b}$ is defined as

$$\bar{a}.\bar{b} = 0 \text{ if one of } \bar{a} \text{ or } \bar{b} \text{ is } 0$$

$$= |\bar{a}||\bar{b}|\text{Cos}\,\theta, \text{ if } \bar{a} \neq 0, \bar{b} \neq 0 \text{ and } \theta \text{ is the angle between } \bar{a} \text{ and } \bar{b}$$

Note:

(i) $\bar{a}\,.\,\bar{b}$ is a scalar.

(ii) If $\bar{a}$, $\bar{b}$ are non zero vectors, than $\bar{a}\,.\,\bar{b}$ is positive or zero or negative according as the angle θ beween $\bar{a}$ and $\bar{b}$ is acute or right or obtuse angle.

(iii) If $\theta = 0^0$, then

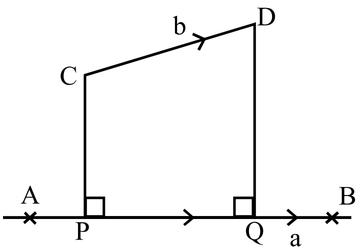
$$\bar{a}.\bar{b} = |\bar{a}||\bar{b}|$$

In particular, $\bar{a}.\bar{a} = |\bar{a}||\bar{a}|\cos 0^o = |\bar{a}|.|\bar{a}|$

$$\bar{a}.\bar{a} = |\bar{a}|^2$$

Orthogonal Projection

Let $\bar{a} = \overline{AB}$ and $\bar{b} = \overline{CD}$ be two non zero vectors. Let P and Q be the feet of the perpendiculars drawn from C and D respectively onto the line AB. Then $\overline{PQ}$ is called the orthogonal projection vector of $\bar{b}$ on $\bar{a}$ and the magnitude, $|\overline{PQ}|$ is called the magnitude of the projection of $\bar{b}$ on $\bar{a}$.



1. The projection vector of $\bar{b}$ on $\bar{a}$ is $\dfrac{(\bar{b}.\bar{a})\bar{a}}{|\bar{a}|^2}$ and its magnitude is $\dfrac{|\bar{b}.\bar{a}|}{|\bar{a}|}$

2. The projection vector of $\bar{a}$ on $\bar{b}$ is $\frac{(\bar{a}.\bar{b})\bar{b}}{|\bar{b}|^2}$ and its magnitude is $\frac{|\bar{a}\,.\,\bar{b}|}{|\bar{b}|}$.

3. Let $\bar{a}$, $\bar{b}$ be two vectors. Then

(i) $\bar{a}.\bar{b} = \bar{b}.\bar{a}$ (commutative law)

(ii) $(l\bar{a})\,.\,\bar{b} = \bar{a}.(l\bar{b}) = l(\bar{a}.\bar{b}),\ l \in R.$

(iii) $(l\bar{a})\,.\,(m\bar{b}) = lm\,(\bar{a}.\bar{b}),\ l, m \in R.$

(iv) $(-\bar{a})\,.\,(\bar{b}) = \bar{a}\cdot(-\bar{b}) = -(\bar{a}\cdot\bar{b})$

(v) $(-\bar{a})\,.\,(-\bar{b}) = \bar{a}\cdot\bar{b}$

Note: If i, j, k are mutually perpandicular unit vectors, then

i.i = j.j = k. k = 1

i.j = j. k = k. i = 0

Theorem: Let $\bar{a} = a_1 i + a_2 j + a_3 \bar{k}$

$\bar{b} = b_1 i + b_2 j + b_3 \bar{k}$. Then

$\bar{a}\cdot\bar{b} = a_1b_1 + a_2b_2 + a_3b_3.$

Note: (i) If θ is the angle between two non-zero vectors $\bar{a}$ and $\bar{b}$, then

$$\theta = \cos^{-1}\left(\frac{\bar{a}.\bar{b}}{|\bar{a}||\bar{b}|}\right)$$

$$\theta = \cos^{-1}\left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}\right)$$

(ii) $\bar{a}$, $\bar{b}$ are perpendicular to each other $\Leftrightarrow a_1b_1 + a_2b_2 + a_3b_3 = 0$

Cross Product of two vectors:

Let $\bar{a}$ and $\bar{b}$ be non-zero non collinear vectors. The cross (or vector) product of $\bar{a}$ and $\bar{b}$ written as $\bar{a} \times \bar{b}$ is defined to be the vector $(|\bar{a}||\bar{b}|\,\text{Sin}\theta)\,\bar{n}$ where θ is the angle between $\bar{a}$ and $\bar{b}$ and $\bar{n}$ is the unit vector perpendicular to both $\bar{a}$ and $\bar{b}$ such that $(\bar{a}, \bar{b}, \bar{n})$ is a right handed system.

If one of the vectors $\bar{a}$, $\bar{b}$ is the null vectors or $\bar{a}$, $\bar{b}$ are collinear vectors then the cross product $\bar{a} \times \bar{b}$ is defined as the null vector $\bar{0}$.

Note:

(1) If $\bar{a}$, $\bar{b}$ are non-zero and non collinear vectors, then $\bar{a} \times \bar{b}$ is a vector, perpendicular to the plane determined by $\bar{a}$ and $\bar{b}$, whose magnitude is $|\bar{a}||\bar{b}|\sin\theta$.

(2) $\bar{a} \times \bar{b} = -(\bar{b} \times \bar{a})$

(3) $(-a) \times \bar{b} = a \times (-\bar{b}) = -(\bar{a} \times \bar{b}) = \bar{b} \times \bar{a}$

(4) $(-\bar{a}) \times (-\bar{b}) = \bar{a} \times \bar{b}$

(5) $(l\bar{a}) \times (\bar{b}) = l(\bar{a} \times \bar{b}) = a \times (l\bar{b}),\ l \in R$

(6) $(l\bar{a}) \times (m\bar{b}) = lm(\bar{a} \times \bar{b}),\ l, m \in R$

(7) $\bar{a} \times (\bar{b} + \bar{c}) = \bar{a} \times \bar{b} + \bar{a} \times \bar{c}$

(8) $(\bar{a} + \bar{b}) \times \bar{c} = (\bar{a} \times \bar{c}) + (\bar{b} \times \bar{c})$

(9) If (i, j, k) is an orthogonal triad, then

(i) $i \times i = j \times j = k \times k = 0$

(ii) $i \times j = k,\ j \times k = i,\ k \times i = j$

Theorem: If $\bar{a} = a_1 i + a_2 j + a_3 k$

If $\bar{b} = b_1 i + b_2 j + b_3 \bar{k}$ then

$$\bar{a} \times \bar{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Theorem: For any two vectors $\bar{a}$ and $\bar{b}$,

$$|\bar{a} \times \bar{b}|^2 = |\bar{a}|^2 |\bar{b}|^2 - (\bar{a} \cdot \bar{b})^2$$

Theorem: The vector area of ΔABC is

$$= \frac{1}{2}\left(\overline{AB} \times \overline{AC}\right) = \frac{1}{2}\left(\overline{BC} \times \overline{BA}\right) = \frac{1}{2}\left(\overline{CA} \times \overline{CB}\right)$$

Theorem: If $\bar{a}$, $\bar{b}$, $\bar{c}$ are the position vectors of the vertices A, B and C of ΔABC, Then the vector area of ΔABC is $\frac{1}{2}\left(\bar{b} \times \bar{c} + \bar{c} \times \bar{a} + \bar{a} \times \bar{b}\right)$ and its area is $\frac{1}{2}\left|\bar{b} \times \bar{c} + \bar{c} \times \bar{a} + \bar{a} \times \bar{b}\right|$

Theorem:

(i) The vector area of any plane quadrilateral ABCD in terms of the diagonals $\overline{AC}$ and $\overline{BD}$ is $\frac{1}{2}\left(\overline{AC} \times \overline{BD}\right)$

(ii) The area of the quadrilateral ABCD is $\frac{1}{2}\left|\overline{AC} \times \overline{BD}\right|$

(iii) The vector area of a parallelogram with $\bar{a}$ and $\bar{b}$ as adjacent sides is $\bar{a} \times \bar{b}$ and the area is $|\bar{a} \times \bar{b}|$,

(iv) The unit vector perpandicular to both $\bar{a}$ and $\bar{b}$ is

$$= \pm\frac{(\bar{a}\times\bar{b})}{|\bar{a}\times\bar{b}|}$$

1. If $\bar{a}$ = 6i + 2j +3k , $\bar{b}$ = 2i – 9j + 6k, then find $\bar{a}\,.\,\bar{b}$ and the angle between $\bar{a}$ and $\bar{b}$. **(4M)**

Sol: $\bar{a}$ = 6i + 2j +3k , $\bar{b}$ = 2i – 9j + 6k then,

$\bar{a}\cdot\bar{b} = a_1b_1 + a_2b_2 + a_3b_3$ and $|\bar{a}| = \sqrt{a_1^{\,2}+a_2^{\,2}+a_3^{\,2}}$

$\bar{a}\cdot\bar{b} = 6(2) + 2(-9) + 3(6) = 12 - 18 + 18 = 12$

$|\bar{a}| = \sqrt{36+4+9} = \sqrt{49} = 7$

$|\bar{b}| = \sqrt{2^2+(-9)^2+6^2} = \sqrt{4+81+36} = \sqrt{121} = 11$

$$\text{Cos}\,\theta = \frac{\bar{a}\,.\,\bar{b}}{\left(|\bar{a}||\bar{b}|\right)} = \frac{12}{7\times 11} = \frac{12}{77}$$

$$\theta = \text{Cos}^{-1}\left(\frac{12}{77}\right)$$

2. If $\bar{a}$ = i + 2j – 3k , $\bar{b}$ = 3i – j + 2k, then show that $\bar{a}+\bar{b}$ and $\bar{a}-\bar{b}$ are perpendicular to each other. **(4 M)**

Sol: $\bar{a}+\bar{b}$ = i + 2j – 3k + 3i – j + 2k = 4i + j – k

$\bar{a}-\bar{b}$ = (i + 2j – 3k) – (3i – j + 2k) = –2i + 3j – 5k

$(\bar{a}+\bar{b})\,.\,(\bar{a}-\bar{b}) = 4(-2) + 1(3) + (-1)(-5)$

$= -8 + 3 + 5$

$= 0$ $\qquad [\because\ \bar{a}\cdot\bar{b} = 0 \Rightarrow \bar{a}\perp\bar{b}]$

$\therefore (\bar{a}+\bar{b}) \perp (\bar{a}-\bar{b})$

3. If $\bar{a}$ = i – j – k and $\bar{b}$ = 2i – 3j + k then find the orthogonal projection of $\bar{b}$ on $\bar{a}$ and its magnitude. **(4 M)**

Sol: Orthogonal projection of $\bar{b}$ on $\bar{a}$ = $\dfrac{(\bar{b}.\bar{a})\bar{a}}{|\bar{a}|^2}$

$\bar{b}\cdot\bar{a}$ = (2i – 3j + k) . (i – j – k)

= 2(1) + (–3)(–1) + 1(–1)= 2 + 3 – 1 = 4

$|\bar{a}| = \sqrt{(1)^2+(-1)^2+(-1)^2} = \sqrt{1+1+1} = \sqrt{3}$

$\therefore$ Orthogonal projection of $\bar{b}$ on $\bar{a}$ $= \dfrac{(\bar{b}.\bar{a})\bar{a}}{|\bar{a}|^2} = \dfrac{4(i-j-k)}{(\sqrt{3})^2} = \dfrac{4(i-j-k)}{3}$

Magnitude of the projection vector $= \dfrac{|\bar{b}.\bar{a}|}{|\bar{a}|} = \dfrac{|4|}{|\sqrt{3}|} = \dfrac{4}{\sqrt{3}}$

4. If the vectors $\lambda i - 3j + 5k$ and $2\lambda i - \lambda j - k$ are perpendicular to each other, find λ. **(2M)**

Sol: If $\bar{a}$ and $\bar{b}$ are perpendicular to each other then, $\bar{a}\cdot\bar{b} = 0$

$\therefore$ $(\lambda)(2\lambda) + (-3)(-\lambda) + 5(-1) = 0$

$2\lambda^2 + 3\lambda - 5 = 0$

$2\lambda^2 + 5\lambda - 2\lambda - 5 = 0$

$(2\lambda + 5)(\lambda - 1) = 0$

$2\lambda + 5 = 0$ (or) $\lambda - 1 = 0$

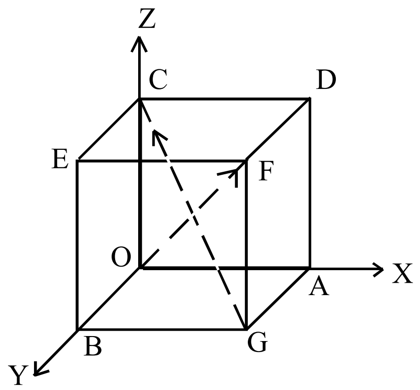
$\lambda = \dfrac{-5}{2}$ (or) $\lambda = 1$

5. Prove that the angle θ between any two diagonals of a cube is given by $\cos\theta = 1/3$.**(4M)**

Sol: Let the cuber be a unit cube.

Let $\overline{OA} = i$; $\overline{OB} = j$; $\overline{OC} = k$

$\overline{OF}$, $\overline{GC}$ are diagonals



$\overline{OF} = OA + AD + DF$

$= i + k + j$

$= i + j + k$

$\overline{GC} = \overline{GB} + \overline{BO} + \overline{OC}$

$= -i - j + k$

If θ is angle between $\overline{OF}$ and $\overline{GC}$, then

$$\text{Cos}\theta = \frac{|\overline{OF}.\overline{GC}|}{|\overline{OF}||\overline{GC}|} = \frac{|1(-1)+1(-1)+1(1)|}{\sqrt{1^2+1^2+1^2}\sqrt{(-1)^2+(-1)^2+1^2}} = \frac{|-1-1+1|}{\sqrt{3}.\sqrt{3}} = \frac{1}{3}$$

6. The Vectors $\overline{AB} = 3i - 2j + 2k$ and $\overline{AD} = i - 2k$ represent the adjacent sides of a parallelogram ABCD, Find the angle between the diagonals. **(4M)**

Sol: $\overline{AC} = \overline{AB} + \overline{BC}$

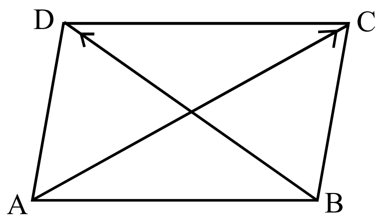
$= 3i - 2j + 2k + i - 2k$

$= 4i - 2j$

$\overline{BD} = BA + AD$

$= -3i + 2j - 2k + i - 2k$

$= -2i + 2j - 4k$



If θ is the angle between $\overline{AC}$ and $\overline{BD}$, then

$$\text{Cos}\theta = \frac{\overline{AC}.\overline{BD}}{|\overline{AC}|\ |\overline{BD}|} = \frac{4(-2)+(-2)2+0(-4)}{\sqrt{4^2+(-2)^2}\sqrt{(-2)^2+2^2+(-4)^2}}$$

$$\text{Cos}\theta = \frac{-8-4}{\sqrt{16+4}\sqrt{4+4+16}} = \frac{-12}{\sqrt{20}\sqrt{24}} = \frac{-12}{\sqrt{5\times4}\sqrt{6\times4}} = \frac{-12}{4\sqrt{30}} = \frac{-3}{\sqrt{10}.\sqrt{3}}$$

$$\text{Cos}\theta = \frac{-\sqrt{3}}{\sqrt{10}}$$

7. Find the cartesian equation of the plane through the point A = (2, –1, –4) and parallel to the plane $4x - 12y - 3z - 7 = 0$. **(4M)**

Sol: The normal to the plane $4x - 12y - 3z - 7 = 0$ is, $\bar{n} = 4i - 12j - 3\bar{k}$.

Let $P = xi + yj + zk$ be any point on the plan.

$\overline{AP} \perp \bar{n}$

$\left(\overline{OP} - \overline{OA}\right).\bar{n} = 0$

$[(x-2)i + (y+1)j + (z+4)\bar{k}] . (4i - 12j - 3\bar{k}) = 0$

$4(x-2) - 12(y+1) - 3(z+4) = 0$

$4x - 12y - 3z - 8 - 12 - 12 = 0$

$4x - 12y - 3z - 32 = 0$

8. Find the angle between the vectors $i + 2j + 3k$ and $3i - j + 2k$. **(4M)**

Sol: Let $\bar{a} = i + 2j + 3k$, $\bar{b} = 3i - j + 2k$

$\bar{a}\cdot\bar{b} = 1(3) + 2(-1) + 3(2) = 3 - 2 + 6 = 7$

$|\bar{a}| = \sqrt{1^2+2^2+3^2} = \sqrt{1+4+9} = \sqrt{14}$

$|\bar{b}| = \sqrt{3^2+(-1)^2+2^2} = \sqrt{9+1+4} = \sqrt{14}$

If θ is angle between $\bar{a}$ and $\bar{b}$ $\Rightarrow$ $\text{Cos}\theta = \dfrac{\bar{a}\ .\ \bar{b}}{|\bar{a}|\ |\bar{b}|}$

$$\text{Cos}\theta = \frac{7}{\sqrt{14}.\sqrt{14}} = \frac{7}{14} = \frac{1}{2} = \text{Cos}60^0$$

$\theta = 60^0$

9. If the vectors $2i + \lambda j - k$ and $4i - 2j + 2k$ are perpendicular to each other, find λ. (2M)

Sol: Let $\bar{a} = 2i + \lambda j - k$; $\bar{b} = 4i - 2j + 2\bar{k}$

If $\bar{a}$ and $\bar{b}$ are perpendicular to each other, then $\bar{a} \cdot \bar{b} = 0$

$\therefore\ 2(4) + \lambda(-2) + (-1)(2) = 0$

$8 - 2\lambda - 2 = 0$

$2\lambda = 6 \quad \therefore \lambda = 3$

10. For what value of λ, the vectors $i - \lambda j + 2k$ and $8i + 6j - k$ are at right angles? **(2M)**

Sol: Let $\bar{a} = i - \lambda j + 2k$; $\bar{b} = 8i + 6j - \bar{k}$

If $\bar{a}$ and $\bar{b}$ are perpendicular to each other, then $\bar{a} . \bar{b} = 0$

$\therefore\ 1(8) + (-\lambda)(6) + 2(-1) = 0$

$\Rightarrow 8 - 6\lambda - 2 = 0$

$\Rightarrow 6\lambda = 6$

$\therefore\ \lambda = 1$

11. Let e_1 and e_2 be unit vectors making angle θ. If $\frac{1}{2}\left|\bar{e}_1 - \bar{e}_2\right| = \text{Sin}\lambda\theta$, then find λ. (4M)

Sol: $\left|\bar{e}_1\right| = 1$; $\left|\bar{e}_2\right| = 1$

$\text{Cos}\theta = \frac{\bar{e}_1 . \bar{e}_2}{\left|\bar{e}_1\right|\left|\bar{e}_2\right|} = \bar{e}_1 . \bar{e}_2$

Given, $\frac{1}{2}\left|\bar{e}_1 - \bar{e}_2\right| = \text{Sin}\lambda\theta$

$\left|\bar{e}_1 - \bar{e}_2\right| = 2\text{Sin}\lambda\theta$

$\left|\bar{e}_1 - \bar{e}_2\right|^2 = 4\text{Sin}^2\lambda\theta \qquad (\because \left|\bar{a}\right|^2 = \bar{a}.\bar{a})$

$\left(\bar{e}_1 - \bar{e}_2\right).\left(\bar{e}_1 - \bar{e}_2\right) = 4\text{Sin}^2\lambda\theta \qquad (\because \bar{e}_1.\bar{e}_1 = \left|\bar{e}_1\right|^2)$

$\left|\bar{e}_1\right|^2 - \bar{e}_1.\bar{e}_2 - \bar{e}_2.\bar{e}_1 + \left|\bar{e}_2\right|^2 = 4\text{Sin}^2\lambda\theta \qquad (\because \bar{e}_1.\bar{e}_2 = \bar{e}_2.\bar{e}_1)$

$1 - 2\ \bar{e}_1.\bar{e}_2 + 1 = 4\text{Sin}^2\lambda\theta$

$2 - 2\text{Cos}\theta = 4\text{Sin}^2\lambda\theta \qquad (\because \bar{e}_1.\bar{e}_2 = \cos\theta)$

$2(1 - \text{Cos}\theta) = 4\text{Sin}^2\lambda\theta$

$2(2\text{Sin}^2\theta/2) = 4\text{Sin}^2\lambda\theta \qquad [\because 1 - \cos\theta = 2\sin^2\frac{\theta}{2}]$

$\text{Sin}^2\theta/2 = \text{Sin}^2\lambda\theta$

$\Rightarrow \lambda = \frac{1}{2}$

12. If $\bar{a} = 2i + 2j - 3k$, $\bar{b} = 3i - j + 2k$, then find the angle betwen the vectors, $2\bar{a} + \bar{b}$ and $\bar{a} + 2\bar{b}$. **(4M)**

Sol: $2\bar{a} + \bar{b} = 2(2i + 2j - 3k) + 3i - j + 2k = 7i + 3j - 4k$

$\bar{a} + 2\bar{b} = 2i + 2j - 3k + 2(3i - j + 2k) = 8i + k$

If θ is the angle between $2\bar{a} + \bar{b}$ and $\bar{a} + 2\bar{b}$, then

$$\text{Cos}\theta = \frac{(2\bar{a}+\bar{b})\bullet(\bar{a}+2\bar{b})}{|2\bar{a}+\bar{b}||\bar{a}+2\bar{b}|} = \frac{7(8)+3(0)+(-4)(1)}{\sqrt{7^2+3^2+(-4)^2}.\sqrt{8^2+1^2}}$$

$$= \frac{56-4}{\sqrt{49+9+16}.\sqrt{64+1}} = \frac{52}{\sqrt{74}.\sqrt{65}}$$

$$\theta = \text{Cos}^{-1}\left(\frac{52}{\sqrt{74}.\sqrt{65}}\right)$$

13. If $\bar{a} + \bar{b} + \bar{c} = 0$, $|\bar{a}| = 3$, $|\bar{b}| = 5$, $|\bar{c}| = 7$, then find the angle between $\bar{a}$ and $\bar{b}$. **(4M)**

Sol: $\bar{a} + \bar{b} = -\bar{c}$

$(\bar{a} + \bar{b})^2 = (-\bar{c})^2$

$(\bar{a} + \bar{b}).(\bar{a} + \bar{b}) = \bar{c}.\bar{c}$

$$|\bar{a}|^2 + \bar{a}\cdot\bar{b} + \bar{b}\cdot\bar{a} + |\bar{b}|^2 = |\bar{c}|^2$$

$$|\bar{a}|^2 + 2\bar{a}\cdot\bar{b} + |\bar{b}|^2 = |\bar{c}|^2$$

$$2|\bar{a}||\bar{b}|\cos\theta = |\bar{c}|^2 - |\bar{a}|^2 - |\bar{b}|^2$$

$2(3)(5)\cos\theta = 49 - 9 - 25$

$30\cos\theta = 49 - 34 = 15$

$$\text{Cos}\theta = \frac{15}{30}$$

$$\text{Cos}\theta = \frac{1}{2} = \text{Cos}60^0$$

$\theta = 60^0$

14. If $|a| = 2$, $|\bar{b}| = 3$ and $|\bar{c}| = 4$ and each of $\bar{a}$, $\bar{b}$, $\bar{c}$ is perpendicular to the sum of the other two vectos, then find the magnitude of $\bar{a} + \bar{b} + \bar{c}$. **(4M)**

Sol: Given $|a| = 2$, $|\bar{b}| = 3$ and $|\bar{c}| = 4$

$\bar{a} \perp (\bar{b}+\bar{c}) \Rightarrow \bar{a}.(\bar{b}+\bar{c}) = 0$

$\bar{a}.\bar{b} + \bar{a}.\bar{c} = 0$

$\bar{b} \perp (\bar{c}+\bar{a}) \Rightarrow \bar{b}.(\bar{c}+\bar{a}) = 0$

$\bar{b}.\bar{c}+\bar{b}.\bar{a} = 0$

$\bar{c} \perp (\bar{a}+\bar{b}) \Rightarrow \bar{c}.(\bar{a}+\bar{b}) = 0$

$\bar{c}.\bar{a} + \bar{c}.\bar{b} = 0$

$\bar{a}.\bar{b} + \bar{a}.\bar{c} + \bar{b}.\bar{c}+\bar{b}.\bar{a} + \bar{c}.\bar{a} + \bar{c}.\bar{b} = 0$

$2(\bar{a}.\bar{b} + \bar{b}.\bar{c} + \bar{c}.\bar{a}) = 0$ ....(1)

Now, $|\bar{a}+\bar{b}+\bar{c}|^2 = (\bar{a}+\bar{b}+\bar{b}).(\bar{a}+\bar{b}+\bar{c})$

$= |\bar{a}|^2 + \bar{a}.\bar{b} + \bar{a}.\bar{c} + \bar{b}.\bar{a} + |\bar{b}|^2 + \bar{b}.\bar{c} + \bar{c}.\bar{a} + \bar{c}.\bar{b} + |\bar{c}|^2$

$= |\bar{a}|^2 + |\bar{b}|^2 + |\bar{c}|^2 + 2(\bar{a}.\bar{b} + \bar{b}.\bar{c} + \bar{c}.\bar{a})$

$= 2^2 + 3^2 + 4^2 + 2(0)$ [$\because$ from (1)]

$= 4 + 9 + 16$

$= 29$

$\therefore |\bar{a}+\bar{b}+\bar{c}| = \sqrt{29}$

15. Show that the points (5, –1, 1), (7, –4, 7) (1, –6, 10) and (–1, –3, 4) are the vertices of a rhombus. **(7M)**

Sol: Let OA = 5i – j + k

OB = 7i – 4j + 7k

OC = i – 6j + 10k

OD = –i – 3j + 4k

$\overline{AB} = \overline{OB} - \overline{OA}$ = 2i – 3j + 6k

$\overline{BD} = \overline{OD} - \overline{OB}$ = –8i + j – 3k

$\overline{AC} = \overline{OC} - \overline{OA}$ = –4i – 5j + 9k

BC = OC – OB = –6i – 2j + 3k

CD = OD – OC = –2i + 3j – 6k

DA = OA – OD = 6i + 2j – 3k

$|\overline{AB}| = \sqrt{4+9+36} = 7$

$|\overline{BC}| = \sqrt{36+4+9} = 7$

$|\overline{CD}| = \sqrt{4+9+36} = 7$

$|\overline{DA}| = \sqrt{36+4+9} = 7$

$|\overline{BD}| = \sqrt{64+1+9} = \sqrt{74}$

$|\overline{AC}| = \sqrt{16+25+81} = \sqrt{122}$

i.e, $|\overline{AB}| = |\overline{BC}| = |\overline{CD}| = |\overline{DA}| \& |\overline{BD}| \neq |\overline{AC}|$

ABCD is a rhombus.

16. If $\bar{a} = 2i - 3j + 5k$, $\bar{b} = -i + 4j + 2k$, then find $\bar{a} \times \bar{b}$ and unit vector perpendicular to both $\bar{a}$ and $\bar{b}$. **(4M)**

Sol: $\bar{a} \times \bar{b} = \begin{vmatrix} i & j & k \\ 2 & -3 & 5 \\ -1 & 4 & 2 \end{vmatrix}$

$= i\begin{vmatrix} -3 & 5 \\ 4 & 2 \end{vmatrix} - j\begin{vmatrix} 2 & 5 \\ -1 & 2 \end{vmatrix} + k\begin{vmatrix} 2 & -3 \\ -1 & 4 \end{vmatrix}$

$= i(-6 - 20) - j(4 + 5) + k(8 - 3)$

$= -26i - 9j + 5k$

Unit vector perpendicular to both $\bar{a}$ and $\bar{b}$ is

$$= \pm\frac{\bar{a} \times \bar{b}}{|\bar{a} \times \bar{b}|}$$

$$= \pm\frac{(-26i - 9j + 5k)}{\sqrt{(-26)^2 + (-9)^2 + 5^2}} = \pm\frac{(-26i - 9j + 5k)}{\sqrt{782}}$$

17. If $a = 2i - 3j + 5k$, $b = -i + 4j + 2k$, then find $(a + b) \times (\vec{a} - \vec{b})$ and unit vector perpendicular to both $\bar{a} + \bar{b}$ and $\bar{a} - \bar{b}$. **(4M)**

Sol: $\bar{a} + \bar{b} = i + j + 7k$, $\quad \bar{a} - \bar{b} = 3i - 7j + 3k$

$(\bar{a} + \bar{b}) \times (\bar{a} - \bar{b}) = \begin{vmatrix} i & j & k \\ 1 & 1 & 7 \\ 3 & -7 & 3 \end{vmatrix}$

$= i(3 + 49) - j(3 - 21) + k(-7 - 3)$

$= 52i + 18j - 10k$

$|(\bar{a} + \bar{b}) \times (\bar{a} - \bar{b})| = \sqrt{(52)^2 + (18)^2 + (-10)^2} = \sqrt{4[(26)^2 + (9)^2 + 5^2]} = 2\sqrt{782}$

Unit vector perpendicular to both $\bar{a} + \bar{b}$ and $\bar{a} - \bar{b}$

$$= \pm\frac{(\bar{a} + \bar{b}) \times (\bar{a} - \bar{b})}{|(\bar{a} + \bar{b}) \times (\bar{a} - \bar{b})|}$$

$$= \pm\frac{(52i + 18j - 10k)}{2\sqrt{782}} = \pm\frac{(26i + 9j - 5k)}{\sqrt{782}}$$

18. Find the area of the parallelogram for which $\bar{a} = 2i - 3j$, $\bar{b} = 3i - k$ are adjacent sides. **(2M)**

Sol: $\bar{a} = 2i - 3j$, $\bar{b} = 3i - k$

Vector area of parallelogram = $\bar{a} \times \bar{b} = \begin{vmatrix} i & j & k \\ 2 & -3 & 0 \\ 3 & 0 & -1 \end{vmatrix}$

$= i(3 - 0) - j(-2 - 0) + k(0 + 9)$

$= 3i + 2j + 9k$

Area = $|\bar{a} \times \bar{b}| = \sqrt{3^2 + 2^2 + 9^2}$

$= \sqrt{9+4+81}$

$= \sqrt{94}$

19. If $\bar{a} = i + 2j + 3k$ and $\bar{b} = 3i + 5j - k$ are two sides of a triangle, then find its area. **(4M)**

Sol: Area of triangle $\frac{1}{2}|\bar{a} \times \bar{b}|$

$\bar{a} \times \bar{b} = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 3 & 5 & -1 \end{vmatrix}$

$= i(-2 - 15) - j(-1 - 9) + k(5 - 6)$

$= -17i + 10j - k$

$|\bar{a} \times \bar{b}| = \sqrt{(-17)^2 + (10^2 + (-1)^2}$

$= \sqrt{289 + 100 + 1}$

$= \sqrt{390}$

$\therefore$ Area of triangle $= \frac{1}{2}|\bar{a} \times \bar{b}| = \frac{1}{2}|\sqrt{390}|$

$= \frac{\sqrt{390}}{2}$

20. If θ is the angle between $a = 2i - j + k$ and $b = 3i + 4j - k$, then find $\sin\theta$. **(4M)**

Sol: $\text{Sin}\theta = \frac{|\bar{a} \times \bar{b}|}{|\bar{a}|\ |\bar{b}|}$

$$\bar{a}\times\bar{b} = \begin{vmatrix} i & j & k \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix}$$

$= i(1-4) - j(-2-3) + k(8+3)$

$= -3i + 5j + 11k$

$$|\bar{a} \times \bar{b}| = \sqrt{(-3)^2 + 5^2 + 11^2} = \sqrt{9+15+121} = \sqrt{155}$$

$$|\bar{a}| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{4+1+1} = \sqrt{6}$$

$$|\bar{b}| = \sqrt{3^2 + 4^2 + (-1)^2} = \sqrt{9+16+1} = \sqrt{26}$$

$$\therefore \text{Sin}\theta = \frac{\sqrt{155}}{\sqrt{6}.\sqrt{26}} = \frac{\sqrt{155}}{\sqrt{156}}$$

21. Let $\bar{a} = 2i + j - 2k$, $\bar{b} = i + j$. If $\bar{c}$ is a vector such that $\bar{a}.\bar{c} = |\bar{c}|$, $|\bar{c} - \bar{a}| = 2\sqrt{2}$ and the angle between $\bar{a}\times\bar{b}$ and $\bar{c}$ is 30º, then find the value of $|(\bar{a}\times\bar{b})\times\bar{c}|$. **(4M)**

Sol: $|\bar{a}| = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{4+1+4} = 3$

$|\bar{b}| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$|\bar{c} - \bar{a}| = 2\sqrt{2}$

$|\bar{c} - \bar{a}|^2 = \left(2\sqrt{2}\right)^2$

$|\bar{c}|^2 + |\bar{a}|^2 - 2(\bar{c}\,.\,\bar{a}) = 8$

$|\bar{c}|^2 + 9 - 2|\bar{c}| = 8$

$|\bar{c}|^2 - 2|\bar{c}| + 1 = 0$

$\left(|\bar{c}| - 1\right)^2 = 0$

$|\bar{c}| = 1$

$|(\bar{a}\times\bar{b})\times\bar{c}| = |\bar{a}\times\bar{b}||\bar{c}|.\text{Sin}30^0$

$= |\bar{a}\times\bar{b}|(1)\frac{1}{2}$

$|(\bar{a}\times\bar{b})\times\bar{c}| = \frac{1}{2}|\bar{a}\times\bar{b}|$ ....(1)

$$\bar{a}\times\bar{b} = \begin{vmatrix} i & j & k \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix}$$

$= i(0 + 2) – j(0 + 2) + k(2 – 1)$

$= 2i – 2j + k$

$\left|\bar{a}\times\bar{b}\right| = \sqrt{4+4+1} = 3$

(1) $\Rightarrow \left|\left(\bar{a}\times\bar{b}\right)\times\bar{c}\right| = \frac{1}{2}(3) = \frac{3}{2}$

22. Let $\bar{a} = 4i + 5j – k$, $\bar{b} = i – 4j + 5k$ and $\bar{c} = 3i + j – k$. Find vector α which is perpendicular to both $\bar{a}$ and $\bar{b}$ and $\alpha.\ \bar{c} = 21$. (4M)

Sol: There exist scalar λ such that $\bar{\alpha} = \lambda(\bar{a}\times\bar{b})$

$$\bar{a}\times\bar{b} = \begin{vmatrix} i & j & k \\ 4 & 5 & -1 \\ 1 & -4 & 5 \end{vmatrix}$$

$= i(25 – 4) – j(20 + 1) + k(–16 – 5)$

$= 21i – 21j – 21k$

$\therefore\ \bar{\alpha} = \lambda(21i – 21j – 21k)$

$\bar{\alpha} = 21\lambda(i – j – k)$

but $\bar{\alpha}\,.\,\bar{c} = 21$

$21\lambda\ (i – j – k)\ .\ (3i + j – k) = 21$

$21\lambda\ (3 – 1 + 1) = 21$

$21 \times 3 \times \lambda = 21$

$\lambda = \frac{1}{3}$

$\therefore\ \alpha = 21\left(\frac{1}{3}\right)(i – j – k)$

$\alpha = 7(i – j – k) = 7i –7j – 7k$

23. For any vector $\bar{a}$, show that $|\bar{a} \times i|^2 + |\bar{a} \times j|^2 + |\bar{a} \times k|^2 = 2\left|\bar{a}\right|^2$. (4M)

Sol: If $\bar{a} = xi + yj + zk$, then $\left|\bar{a}\right| = \sqrt{x^2 + y^2 + z^2}$

$$\bar{a} \times i = \begin{vmatrix} i & j & k \\ x & y & z \\ 1 & 0 & 0 \end{vmatrix}$$

$= i(0-0) - j(0-z) + k(0-y)$

$= zj - yk$

$|\bar{a} \times i| = \sqrt{z^2 + y^2}$

Similarly $|\bar{a} \times j| = \sqrt{z^2 + x^2}$

$|\bar{a} \times k| = \sqrt{x^2 + y^2}$

$\therefore\ |\bar{a} \times i|^2 + |\bar{a} \times j|^2 + |\bar{a} \times k|^2$

$= z^2 + y^2 + z^2 + x^2 + x^2 + y^2 = 2(x^2 + y^2 + z^2)$

$= 2.\left(\sqrt{x^2 + y^2 + z^2}\right)^2$

$= 2|a|^2$

24. If $\bar{a} = 2i - j + k$, $\bar{b} = i - 3j - 5k$, then find $|\bar{a} \times \bar{b}|$. (2M)

Sol: $\bar{a} \times \bar{b} = \begin{vmatrix} i & j & k \\ 2 & -1 & 1 \\ 1 & -3 & -5 \end{vmatrix}$

$= i(5+3) - j(-10-1) + k(-6+1)$

$= 8i + 11j - 5k$

$\therefore\ |\bar{a} \times \bar{b}| = \sqrt{8^2 + 11^2 + (-5)^2}$

$= \sqrt{64 + 121 + 25}$

$= \sqrt{210}$

25. If $\bar{a} = 2i - 3j + k$, $\bar{b} = i + 4j - 2k$, then find $(\bar{a} + \bar{b}) \times (\bar{a} - \bar{b})$. (2M)

Sol: $\bar{a} + \bar{b} = 3i + j - k$

$\bar{a} - \bar{b} = i - 7j + 3k$

$(\bar{a} + \bar{b}) \times (\bar{a} - \bar{b}) = \begin{vmatrix} i & j & k \\ 3 & 1 & -1 \\ 1 & -7 & 3 \end{vmatrix}$

$= i(3-7) - j(9+1) + k(-21-1)$

$= -4i - 10j - 22k$

26. If $4i + \frac{2p}{3}j + pk$ is parallel to the vector $i + 2j + 3k$, find p. (2M)

Sol: If $\bar{a} = a_1 i + a_2 j + a_3 k$ is parallel to $\bar{b} = b_1 i + b_2 j + b_3 k$, then

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

$$\therefore \frac{4}{1} = \frac{2p/3}{2} = \frac{p}{3}$$

$$4 = \frac{p}{3}$$

$\Rightarrow p = 12$

27. Find unit vector perpendicular to both $i + j + k$ and $2i + j + 3k$. (2M)

Sol: The unit vector perpendicular to both $\bar{a}$ and $\bar{b}$ is $= \pm\frac{(\bar{a}\times\bar{b})}{|\bar{a}\times\bar{b}|}$

$$\bar{a}\times\bar{b} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{vmatrix}$$

$= i(3 - 1) - j(3 - 2) + k(1 - 2)$

$= 2i - j - k$

$\therefore |\bar{a} \times \bar{b}| = \sqrt{2^2 + (-1)^2 + (-1)^2}$

$= \sqrt{4+1+1} = \sqrt{6}$

$\therefore$ Required unit vector $= \pm\frac{(2i - j - k)}{\sqrt{6}}$

28. Find the area of the parallelogram having $\bar{a} = 2j - k$ and $\bar{b} = -i + k$ as adjacent sides. (2M)

Sol: Area of parallelogram = $|\bar{a} \times \bar{b}|$

$$\bar{a}\times\bar{b} = \begin{vmatrix} i & j & k \\ 0 & 2 & -1 \\ -1 & 0 & 1 \end{vmatrix}$$

$= i(2 - 0) - j(0 - 1) + k(0 + 2)$

$= 2i + j + 2k$

$\therefore$ Area of parallelogram = $|\bar{a} \times \bar{b}| = \sqrt{2^2 + 1^2 + 2^2}$

$= \sqrt{4+1+4} = 3$

29. Find the area of the triangle whose vertices are A(1, 2, 3), B(2, 3, 1) and C(3, 1, 2). (4M)

Sol: $\overline{OA} = i + 2j + 3k$

$\overline{OB} = 2i + 3j + k$

$\overline{OC} = 3i + j + 2k$

$\overline{AB} = \overline{OB} - \overline{OA} = i + j - 2k$

$\overline{AC} = \overline{OC} - \overline{OA} = 2i - j - k$

$$\overline{AB} \times \overline{AC} = \begin{vmatrix} i & j & k \\ 1 & 1 & -2 \\ 2 & -1 & -1 \end{vmatrix}$$

$= i(-1 - 2) - j(-1 + 4) + k(-1 - 2)$
$= -3i - 3j - 3k$

$\left|\overline{AB} \times \overline{AC}\right| = \sqrt{9+9+9} = \sqrt{27} = 3\sqrt{3}$

Area of triangle $= \frac{1}{2}\left|\overline{AB} \times \overline{AC}\right|$

$= \frac{1}{2}\left(3\sqrt{3}\right) = \frac{3\sqrt{3}}{2}$

30. If $\bar{a} = 2i + j - k$, $\bar{b} = -i + 2j - 4k$, $\bar{c} = i + j + k$, then find $\left(\bar{a} \times \bar{b}\right) . \left(\bar{b} \times \bar{c}\right)$. (4M)

Sol: $$\bar{a} \times \bar{b} = \begin{vmatrix} i & j & k \\ 2 & 1 & -1 \\ -1 & 2 & -4 \end{vmatrix}$$

$= i(-4 + 2) - j(-8 - 1) + k(4 + 1)$
$= -2i + 9j + 5k$

$$\bar{b} \times \bar{c} = \begin{vmatrix} i & j & k \\ -1 & 2 & -4 \\ 1 & 1 & 1 \end{vmatrix}$$

$= i(2 + 4) - j(-1 + 4) + k(-1 - 2)$

$= 6i - 3j - 3k$

$\left(\bar{a} \times \bar{b}\right) . \left(\bar{b} \times \bar{c}\right) = (-2i + 9j + 5k) . (6i - 3j - 3k)$

$= (-2)(6) + (9)(-3) + (5)(-3)$

$= -12 - 27 - 15$

$= -54$

31. Find a unit vector perpendicular to the plane determined by the points P(1, –1, 2), Q(2, 0, –1) and R(0, 2, 1). (4M)

Sol: OP = i – j + 2k; OQ = 2i – k; OR = 2j + k

PQ = OQ – OP = i + j – 3k

PR = OR – OP = –i + 3j – k

$$PQ \times PR = \begin{vmatrix} i & j & k \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix}$$

= i(–1 + 9) – j(–1 – 3) + k(3 + 1)

= 8i + 4j + 4k

= 4(2i + j + k)

$|PQ \times PR| = 4\sqrt{4+1+1} = 4\sqrt{6}$

Required unit vector = $\pm\dfrac{(PQ \times PR)}{|PQ \times PR|}$

$= \pm\dfrac{4(2i + j + k)}{4\sqrt{6}}$

$= \pm\dfrac{(2i + j + k)}{\sqrt{6}}$

32. If $|\bar{a}| = 13$, $|\bar{b}| = 5$, $\bar{a}.\bar{b} = 60$, then find $|\bar{a} \times \bar{b}|$. (2M)

Sol: $|\bar{a} \times \bar{b}|^2 = |\bar{a}|^2 \, |\bar{b}|^2 - (\bar{a}.\bar{b})^2$

$= (13)^2(5)^2 - (60)^2$

= 4225 – 3600 = 625

$\Rightarrow |\bar{a} \times \bar{b}| = 25$

33. If $\bar{a} = 2i + 3j + 4k$, $\bar{b} = i + j - k$, $\bar{c} = i - j + k$, then compute $\bar{a} \times (\bar{b} \times \bar{c})$ and verify that it is perpendicular to $\bar{a}$. (4M)

Sol: $$\bar{b} \times \bar{c} = \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

= i(1 –1) – j(1 + 1) + k(–1 – 1)

= –2j – 2k

$$\bar{a} \times (\bar{b} \times \bar{c}) = \begin{vmatrix} i & j & k \\ 2 & 3 & 4 \\ 0 & -2 & -2 \end{vmatrix}$$

$= i(-6 + 8) - j(-4 - 0) + k(-4 - 0)$

$= 2i + 4j - 4k$

$(\bar{a} \times (\bar{b} \times \bar{c})).\ \bar{a} = (2i + 4j - 4k).\ (2i + 3j + 4k)$

$= 2(2) + 4(3) + (-4)(4)$

$= 4 + 12 - 16$

$= 0$

$\therefore\ \bar{a} \times (\bar{b} \times \bar{c})$ is perpendicular to $\bar{a}$.

34. If $\bar{a} = 7i - 2j + 3k$, $\bar{b} = 2i + 8k$ and $\bar{c} = i + j + k$ then compute $\bar{a} \times \bar{b}$, $\bar{a} \times \bar{c}$, $\bar{a} \times (\bar{b} + \bar{c})$. Verify whether the cross product is distributive over the vector addition.**(7M)**

Sol: $\bar{a} \times \bar{b} = \begin{vmatrix} i & j & k \\ 7 & -2 & 3 \\ 2 & 0 & 8 \end{vmatrix}$

$= i(-16 - 0) - j(56 - 6) + k(0 + 4)$

$= -16i - 50j + 4k$

$\bar{a} \times \bar{c} = \begin{vmatrix} i & j & k \\ 7 & -2 & 3 \\ 1 & 1 & 1 \end{vmatrix}$

$= i(-2 - 3) - j(7 - 3) + k(7 + 2)$

$= -5i - 4j + 9k$

$\bar{b} + \bar{c} = 2i + 8k + i + j + k$

$= 3i + j + 9k$

$$\bar{a} \times (\bar{b} + \bar{c}) = \begin{vmatrix} i & j & k \\ 7 & -2 & 3 \\ 3 & 1 & 9 \end{vmatrix}$$

$= i(-18 - 3) - j(63 - 9) + k(7 + 6)$

$= -21i - 54j + 13k$ ....(1)

$\bar{a} \times \bar{b} + \bar{a} \times \bar{c} = -16i - 50j + 4k + (-5i - 4j + 9k)$

$= -21i - 54j + 13k$ ....(2)

From (1) & (2),

$\bar{a} \times (\bar{b} + \bar{c}) = (\bar{a} \times \bar{b}) + (\bar{a} \times \bar{c})$

$\therefore$ Cross product is distributive over the vector addition.

35. If $\bar{a} = i + j + k$, $\bar{c} = j - k$, then find vector $\bar{b}$ such that $\bar{a} \times \bar{b} = \bar{c}$ and $\bar{a}.\bar{b} = 3$. **(7M)**

Sol: Let $\bar{b} = b_1 i + b_2 j + b_3 k$

Given, $\bar{a} \times \bar{b} = \bar{c}$

$$\begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ b_1 & b_2 & b_3 \end{vmatrix} = j - k$$

$i(b_3 - b_2) - j(b_3 - b_1) + k(b_2 - b_1) = j - k$

$\Rightarrow$ $b_3 - b_2 = 0;\ b_1 - b_3 = 1;\ b_2 - b_1 = -1$

Let $b_3 = b_2 = k$

$b_1 - k = 1$ $\qquad$ $k - b_1 = -1$

$b_1 = 1 + k;$ $\qquad$ $b_1 = k + 1$

Given, $\bar{a} . \bar{b} = 3$

$(i + j + k). (b_1 i + b_2 j + b_3 k) = 3$

$b_1 + b_2 + b_3 = 3$

$k + 1 + k + k = 3$

$3k = 2$

$k = \frac{2}{3}$

$\therefore\ b_1 = \frac{2}{3} + 1 = \frac{5}{3},\ \ b_2 = b_3 = k = \frac{2}{3}$

$\therefore\ \bar{b} = b_1 i + b_2 j + b_3 k$

$= \frac{5}{3} i + \frac{2}{3} j + \frac{2}{3} k = \frac{1}{3}(5i + 2j + 2k)$

36. If $\bar{a}, \bar{b}, \bar{c}$ are unit vectors such that $\bar{a}$ is perpendicular to the plan of $\bar{b}, \bar{c}$ and the angle between $\bar{b}$ and $\bar{c}$ is $\frac{\pi}{3}$, then find $|\bar{a} + \bar{b} + \bar{c}|$. **(7M)**

Sol: $|\bar{a}| = |\bar{b}| = |\bar{c}| = 1$

$\bar{a} \perp \bar{b} \Rightarrow \bar{a} \cdot \bar{b} = 0$

$\bar{a} \perp \bar{c} \Rightarrow \bar{a} . \bar{c} = 0$

$$\left|\bar{a}+\bar{b}+\bar{c}\right|^2 = \left|\bar{a}\right|^2+\left|\bar{b}\right|^2+\left|\bar{c}\right|^2 + 2\left(\bar{a}\,.\,\bar{b}+\bar{b}\,.\,\bar{c}+\bar{c}\,.\,\bar{a}\right)$$

$$= 1+1+1+2\left(0+|b||c|\ \mathrm{Cos}\frac{\pi}{3}+0\right)$$

$$= 1+1+1+2\left(1.1.\frac{1}{2}\right)$$

$$= 1+1+1+1$$

$$\left|\bar{a}+\bar{b}+\bar{c}\right|^2 = 4$$

$$\therefore\ \left|\bar{a}+\bar{b}+\bar{c}\right| = 2$$

37. $\bar{a} = 3i - j + 2k$, $\bar{b} = -i + 3j + 2k$, $\bar{c} = 4i + 5j - 2k$, $\bar{d} = i + 3j + 5k$, then compute (i) $\left(\bar{a}\times\bar{b}\right)\times\left(\bar{c}\times\bar{d}\right)$ (ii) $\left(\bar{a}\times\bar{b}\right)\cdot\bar{c}$ - $\left(\bar{a}\times\bar{d}\right).\bar{b}$. **(7M)**

Sol: $$\bar{a}\times\bar{b} = \begin{vmatrix} i & j & k \\ 3 & -1 & 2 \\ -1 & 3 & 2 \end{vmatrix}$$

$$= i(-2-6) - j(6+2) + k(9-1)$$

$$= -8i - 8j + 8k$$

$$\bar{c}\times\bar{d} = \begin{vmatrix} i & j & k \\ 4 & 5 & -2 \\ 1 & 3 & 5 \end{vmatrix}$$

$$= i(25+6) - j(20+2) + k(12-5)$$

$$= 31i - 22j + 7k$$

(i) $$\left(\bar{a}\times\bar{b}\right)\times\left(\bar{c}\times\bar{d}\right) = \begin{vmatrix} i & j & k \\ -8 & -8 & 8 \\ 31 & -22 & 7 \end{vmatrix}$$

$$= i(-56+176) - j(-56-248) + k(176+248)$$

$$= 120i + 304j + 424k$$

$$\left(\bar{a}\times\bar{b}\right)\cdot\bar{c} = (-8i-8j+8k).(4i+5j-2k)$$

$$= (-8)(4) + (-8)(5) + (8)(-2)$$

$$= -32 - 40 - 16$$

$$= -88$$

$$(\bar{a} \times \bar{d}) = \begin{vmatrix} i & j & k \\ 3 & -1 & 2 \\ 1 & 3 & 5 \end{vmatrix}$$

$= i(-5 - 6) - j(15 - 2) + k(9 + 1)$
$= -11i - 13j + 10k$

$(\bar{a} \times \bar{d}) \cdot \bar{b} = (-11i - 13j + 10k). (-i + 3j + 2k)$

$= (-11)(-1) + (-13)(3) + 10(2)$
$= 11 - 39 + 20$
$= -8$

$\therefore (\bar{a} \times \bar{b}) \cdot \bar{c} - (\bar{a} \times \bar{d}).\bar{b} = -88 - (-8)$

$= -88 + 8$

$= -80$

❖❖❖❖❖

Unit **6**

Trignometric Ratios upto Transformations

1. In a right angled triangle ABC, θ is an acute angle. *x* is opposite side, y is an adjacent side, z is hypotenuse, then

$\text{Sin}\theta = \frac{x}{z}$

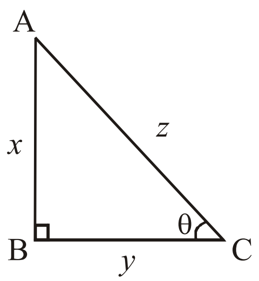
$\text{Cos}\theta = \frac{y}{z}$

$\tan\theta = \frac{x}{y}$

$\text{Cosec}\theta = \frac{z}{x}$

$\text{Sec}\theta = \frac{z}{y}$

$\text{Cot}\theta = \frac{y}{x}$



* From the definitions of trigonometric ratios, we can observe the following

1) $\tan\theta = \frac{\text{Sin}\theta}{\text{Cos}\theta}$ 2) $\text{Cot}\theta = \frac{\text{Cos}\theta}{\text{Sin}\theta}$ 3) $\text{Sec}\theta = \frac{1}{\text{Cos}\theta}$ 4) $\text{Cos}\theta = \frac{1}{\text{Sec}\theta}$

5) $\text{Sin}\theta = \frac{1}{\text{Cosec}\theta}$ 6) $\text{Cosec}\theta = \frac{1}{\text{Sin}\theta}$

Trigonometric Identities

1) $\text{Cos}^2\theta + \text{Sin}^2\theta = 1$
$\text{Cos}^2\theta = 1 - \text{Sin}^2\theta$
$\text{Sin}^2\theta = 1 - \text{Cos}^2\theta$

2) $\text{Sec}^2\theta - \tan^2\theta = 1$
$\text{Sec}^2\theta = 1 + \tan^2\theta$
$\tan^2\theta = \text{Sec}^2\theta - 1$

3) $Cosec^2\theta - Cot^2\theta = 1$

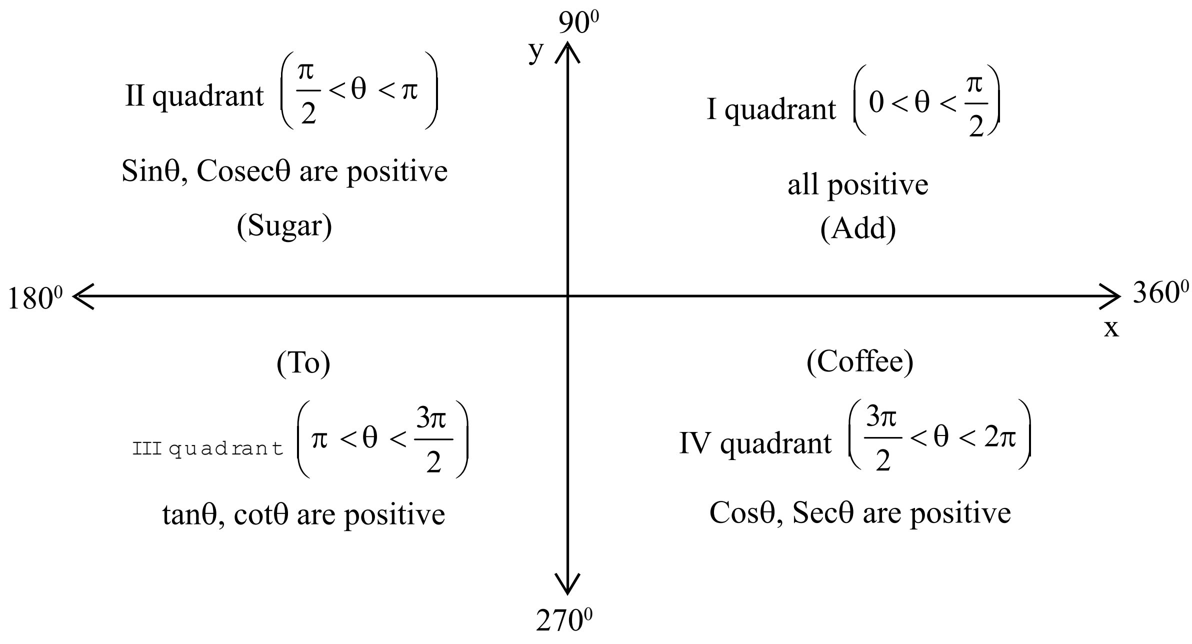
$Cosec^2\theta = 1 + Cot^2\theta$

$Cot^2\theta = Cosec^2\theta - 1$

Values of the Trigonometric Functions

Angle (θ)	0^0	$\frac{\pi}{6} = 30^0$	$\frac{\pi}{4} = 45^0$	$\frac{\pi}{3} = 60^0$	$\frac{\pi}{2} = 90^0$
sinθ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cosθ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tanθ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
cotθ	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
cosecθ	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
secθ	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞

* We can remember the sign of trigonometric functions in four quadrants by using the following figure.



Add	Sugar	To	Coffee
All Trignometric functions are +ve	Sin, Cosec $\}+ve$	tan, Cot $\}+ve$	Cos, Sec $\}+ve$
	Cos, tan, Cot, Sec $\}-ve$	Sin, Cos, Sec, Cosec $\}-ve$	Sin, Cosec, tan, Cot $\}-ve$

Angle (α)	Sinα	Cosα	tanα
$n\pi - \theta$	$(-1)^{n+1}\text{Sin}\theta$	$(-1)^n\text{Cos}\theta$	$-\tan\theta$
$n\pi + \theta$	$(-1)^n\text{Sin}\theta$	$(-1)^n\text{Cos}\theta$	$\tan\theta$
$(2n+1)\frac{\pi}{2} - \theta$	$(-1)^n\text{Cos}\theta$	$(-1)^n\text{Sin}\theta$	$\text{Cot}\theta$
$(2n+1)\frac{\pi}{2} + \theta$	$(-1)^n\text{Cos}\theta$	$(-1)^{n+1}\text{Sin}\theta$	$-\text{Cot}\theta$

* Any trigonometric function for the angle $\frac{n\pi}{2} \pm \theta$ $(n \in Z)$,

(i) If 'n' is even integer, then there is no change in trigonometric function.

(ii) If 'n' is odd integer, then there is change in trigonometric function as follows

Sin $\rightleftharpoons$ Cos tan $\rightleftharpoons$ Cot Sec $\rightleftharpoons$ Cosec

* Sin$(-\theta)$ = –Sinθ, Cos$(-\theta)$ = Cosθ; tan $(-\theta)$ = –tanθ

Cot$(-\theta)$ = –Cotθ, Sec$(-\theta)$ = Secθ; Cosec$(-\theta)$ = – Cosecθ

* All trigonometric functions are periodic functions.

Period of Sinx is 2π

Period of Cosx is 2π

Period of tanx is π

* Range of Sinθ (or) Cosθ is $[-1, 1]$

Range of tanθ (or) Cotθ is R

Range of Secθ (or) Cosecθ is $(-\infty, -1] \cup [1, \infty)$

Compound Angles

* **A, B are any two angles, then**

i) Sin(A + B) = SinACosB + CosASinB

ii) Sin(A – B) = SinACosB – CosASinB

iii) Cos(A + B) = CosACosB – SinASinB

iv) Cos(A – B) = CosACosB + SinASinB

* If none of A, B, A+B, A–B is an odd multiple of $\frac{\pi}{2}$, then

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

* If none of A, B, A+B, A–B is an integral multiple of π, then

$$\cot(A + B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}$$

$$\cot(A - B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$$

* If A, B, C $\in$ R then

$\sin(A+B+C) = \sum(\sin A\cos B\cos C) - \sin A\sin B\sin C$

$\cos(A + B + C) = \cos A\cos B\cos C - \sum(\cos A\sin B\sin C)$

Trigonometric ratios of multiple and sub multiple angles

1. $\sin 2\theta = 2\sin\theta\cos\theta$, $\sin\theta = 2\sin\theta/2\cos\theta/2$

$$= \frac{2\tan\theta}{1+\tan^2\theta} \qquad = \frac{2\tan\theta/2}{1+\tan^2\theta/2}$$

2. $\cos 2\theta = \cos^2\theta - \sin^2\theta$ $\cos\theta = \cos^2\theta/2 - \sin^2\theta/2$

$= 1 - 2\sin^2\theta$ $= 1 - 2\sin^2\theta/2$

$= 2\cos^2\theta - 1$ $= 2\cos^2\theta/2 - 1$

$$= \frac{1-\tan^2\theta}{1+\tan^2\theta} \qquad = \frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}$$

3. $$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} \qquad \tan\theta = \frac{2\tan\theta/2}{1-\tan^2\theta/2}$$

4. $\text{Cot}2\theta = \dfrac{\text{Cot}^2\theta - 1}{2\text{Cot}\theta}$ $\qquad$ $\text{Cot}\theta = \dfrac{\text{Cot}^2 \theta/2 - 1}{2\text{Cot}\,\theta/2}$

5. $1 + \text{Cos}2\theta = 2\text{Cos}^2\theta$ $\qquad$ $1 + \text{Cos}\theta = 2\text{Cos}^2 \theta/2$

6. $1 - \text{Cos}2\theta = 2\text{Sin}^2\theta$ $\qquad$ $1 - \text{Cos}\theta = 2\ \text{Sin}^2\theta/2$

7. $\text{Sin}\theta = \pm\sqrt{\dfrac{1-Cos2\theta}{2}}$ $\qquad$ $\text{Sin}\theta/2 = \pm\sqrt{\dfrac{1-Cos\theta}{2}}$

8. $\text{Cos}\theta = \pm\sqrt{\dfrac{1+Cos2\theta}{2}}$ $\qquad$ $\text{Cos}\theta/2 = \pm\sqrt{\dfrac{1+Cos\theta}{2}}$

9. $\tan\theta = \pm\sqrt{\dfrac{1-Cos2\theta}{1+Cos2\theta}}$ $\qquad$ $\tan\theta/2 = \pm\sqrt{\dfrac{1-Cos\theta}{1+Cos\theta}}$

* $\text{Sin}3\theta = 3\text{Sin}\theta - 4\text{Sin}^3\theta$

$\text{Cos}3\theta = 4\text{Cos}^3\theta - 3\text{Cos}\theta$

$\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

$\text{Cot}3\theta = \dfrac{3Cot\theta - Cot^3\theta}{1-3Cot^2\theta}$

Transformations

I.

* $\text{Sin}(A + B) + \text{Sin}(A - B) = 2\text{SinACosB}$
* $\text{Sin}(A + B) - \text{Sin}(A - B) = 2\text{CosASinB}$
* $\text{Cos}(A + B) + \text{Cos}(A - B) = 2\text{CosACosB}$
* $\text{Cos}(A - B) - \text{Cos}(A + B) = 2\text{SinASinB}$

II.

* $\text{SinC} + \text{SinD} = 2\text{Sin}\left(\dfrac{C + D}{2}\right)\text{Cos}\left(\dfrac{C - D}{2}\right)$
* $\text{SinC} - \text{SinD} = 2\text{Cos}\left(\dfrac{C + D}{2}\right)\text{Sin}\left(\dfrac{C - D}{2}\right)$
* $\text{CosC} + \text{CosD} = 2\text{Cos}\left(\dfrac{C + D}{2}\right)\text{Cos}\left(\dfrac{C - D}{2}\right)$
* $\text{CosC} - \text{CosD} = -2\text{Sin}\left(\dfrac{C + D}{2}\right)\text{Sin}\left(\dfrac{C - D}{2}\right)$

Some Important Problmes with Solutions

1. Simplify the following problems

i. $\text{Cos}315^0 = \text{Cos}(360^0 - 45^0) = \text{Cos}45^0 = \dfrac{1}{\sqrt{2}}$

ii. $\text{Cot}(-300^0) = -\text{Cot}300^0 = -\text{Cot}(360 - 60^0) = -\text{Cot}(-60^0) = \dfrac{1}{\sqrt{3}}$

iii. $\text{Sin}\left(\dfrac{5\pi}{3}\right) = \text{Sin}\left(2\pi - \dfrac{\pi}{3}\right) = -\text{Sin}\dfrac{\pi}{3} = -\dfrac{\sqrt{3}}{2}$

2. Find the value of $\text{Cos}^2 45^0 + \text{Cos}^2 135^0 + \text{Cos}^2 225^0 + \text{Cos}^2 315^0$.

Sol: $\text{Cos}^2 45^0 + \text{Cos}^2 135^0 + \text{Cos}^2 225^0 + \text{Cos}^2 315^0$

$= \text{Cos}^2 45^0 + \text{Cos}^2(180-45^0) + \text{Cos}^2(180+45^0) + \text{Cos}^2(360-45^0)$

$= \text{Cos}^2 45^0 + \text{Cos}^2 45^0 + \text{Cos}^2 45^0 + \text{Cos}^2 45^0$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 2$$

3. Find the value of $\text{Cot}\dfrac{\pi}{20} . \text{Cot}\dfrac{3\pi}{20} . \text{Cot}\dfrac{5\pi}{20} . \text{Cot}\dfrac{7\pi}{20} . \text{Cot}\dfrac{9\pi}{20}$.

Sol: $\text{Cot}\dfrac{\pi}{20} . \text{Cot}\dfrac{3\pi}{20} . \text{Cot}\dfrac{5\pi}{20} . \text{Cot}\dfrac{7\pi}{20} . \text{Cot}\dfrac{9\pi}{20} = \text{Cot}9^0. \text{Cot}27^0. \text{Cot}45^0.\text{Cot}63^0.\text{Cot}81^0$

$\text{Cot}9^0.\text{Cot}27^0.\text{Cot}(90^0 - 27). \text{Cot}(90^0 - 9^0)$

$= \text{Cot}9^0.\text{Cot}27^0.\tan 27^0.\tan 9^0 = 1$

4. Find the value of $\text{Sin}330^0.\text{Cos}120^0 + \text{Cos}210^0.\text{Sin}300^0$.

Sol: $\text{Sin}330^0.\text{Cos}120^0 + \text{Cos}210^0.\text{Sin}300^0$

$= \text{Sin}(360 - 30^0)\ \text{Cos}(180 - 60^0) + \text{Cos}(180 + 30^0)\ \text{Sin}(360 - 60^0)$

$= (-\text{Sin}30^0)\ (-\text{Cos}60^0) + (-\text{Cos}30^0)(-\text{Sin}60^0)$

$$= \frac{1}{2} . \frac{1}{2} + \frac{\sqrt{3}}{2} . \frac{\sqrt{3}}{2} = \frac{1}{4} + \frac{3}{4} = 1$$

5. If $\text{Sin}\alpha + \text{Cosec}\alpha = 2$, $n \in Z$, then find the value of $\text{Sin}^n\alpha + \text{Cosec}^n\alpha$.

Sol: $\text{Sin}\alpha + \text{Coses}\alpha = 2$

$$\text{Sin}\,\alpha + \frac{1}{\text{Sin}\alpha} = 2$$

$\Rightarrow \quad \text{Sin}^2\alpha + 1 = 2\text{Sin}\alpha$

$\Rightarrow \quad 1 - 2\,Sin^2\alpha + Sin^2\alpha = 0$

$\Rightarrow \quad (1 - Sin\alpha)^2 = 0$

$\Rightarrow \quad 1 - Sin\alpha = 0$

$\Rightarrow \quad Sin\alpha = 1 \Rightarrow \alpha = 90^0$

$\therefore\ Sin^n\alpha + Cosec^n\alpha = Sin^n 90^0 + Cosec^n 90^0 = 1^n + 1^n = 1 + 1 = 2$

6. Eliminate θ from the following.

(i) $x = a\,Cos^3\theta$; $y = bSin^3\theta$

Sol: $\frac{x}{a} = Cos^3\theta \qquad \frac{y}{b} = Sin^3\theta$

$Cos\theta = \left(\frac{x}{a}\right)^{1/3} \qquad Sin\theta = \left(\frac{y}{b}\right)^{1/3}$

$\because\ Cos^2\theta + Sin^2\theta = 1$

$\Rightarrow \left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$

ii. $x = a(Sec\theta + tan\theta)$; $y = b(Sec\theta - tan\theta)$

Sol: $xy = ab(Sec^2\theta - tan^2\theta)$

$= ab(1)$

$xy = ab$

7. Find the period of the following functions.

i) $Cos(3x + 5) + 7$

Sol: $f(x) = Cos(3x + 5) + 7$

Period = $\frac{p}{|a|} = \frac{2\pi}{3}$

ii) tan5x

Sol: $f(x) = tan5x$

Period = $\frac{\pi}{5}$

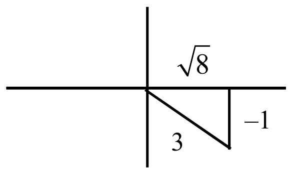
iii) $Cos\left(\frac{4x+9}{5}\right)$

Sol: $f(x) = Cos\left(\frac{4x+9}{5}\right)$

Period $= \frac{2\pi}{4/5} = \frac{10\pi}{4} = \frac{5\pi}{2}$

8. θ is not in 3rd quadrant, if Sinθ = $-\frac{1}{3}$ then find the values of a) Cosθ b) Cotθ.

Sol: $\text{Sin}\theta = -\frac{1}{3} < 0; \theta \notin Q_3$.



$\Rightarrow \theta \in Q_4$.

a) $\text{Cos}\theta = \frac{\sqrt{8}}{3}$ b) $\text{Cot}\theta = -\sqrt{8}$

9. Find the value of $Sin^2 82\frac{1}{2}^0 - Sin^2 22\frac{1}{2}^0$.

Sol: $Sin^2 82\frac{1}{2}^0 - Sin^2 22\frac{1}{2}^0 = \text{Sin}^2\text{A} - \text{Sin}^2\text{B}$

= Sin(A + B) Sin(A – B)

= $\text{Sin}105^0 . \text{Sin}60^0$ [$\because$ substituting $A = 82\frac{1}{2}^0$; $B = 22\frac{1}{2}^0$]

= $\text{Sin}(90 + 15^0)\ \text{Sin}60^0$

= $\text{Cos}15^0 . \text{Sin}60^0$

$= \frac{\sqrt{3}+1}{2\sqrt{2}} . \frac{\sqrt{3}}{2}$

$= \frac{3+\sqrt{3}}{4\sqrt{2}}$

10. Find the value of $\text{Cos}^2 112\frac{1}{2}^0 - \text{Sin}^2 52\frac{1}{2}^0$

Sol: Let $A = 112\frac{1}{2}^0$; $B = 52\frac{1}{2}^0$

$\text{Cos}^2 112\frac{1}{2}^0 - \text{Sin}^2 52\frac{1}{2}^0 = \text{Cos}^2\text{A} - \text{Sin}^2\text{B}$

= Cos(A + B).Cos(A–B) $\Rightarrow$ $\text{Cos}(165^0).\text{Cos}60^0$

= $\text{Cos}(180 - 15^0) . \text{Cos}60^0$

= $-\text{Cos}15^0.\text{Cos}60^0$

$= -\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)\left(\frac{1}{2}\right)$

$= -\left(\frac{\sqrt{3}+1}{4\sqrt{2}}\right)$

11. Find the minimum and maximum value of the function 3cosx + 4sinx.

Sol: Let f(x) = 3cosx + 4sinx

Comparing with f(x) = a sinx + b cosx + c, we get a = 4, b = 3, c = 0.

Minimum value = $c - \sqrt{a^2+b^2}$

$= 0 - \sqrt{4^2+3^2}$

$= -\sqrt{25}$

$= -5$

Maximum value = $c + \sqrt{a^2+b^2}$

$= 0 + \sqrt{4^2+3^2}$

$= \sqrt{25}$

$= 5$

12. Find the minimum and maximum value of the function Sin2x – Cos2x.

Sol: Let f(x) = Sin2x – Cos2x

Comparing with f(x) = a sinx + b cosx + c, we get a = 1, b = –1, c = 0.

Minimum value = $c - \sqrt{a^2+b^2}$

$= -\sqrt{1^2+(-1)^2}$

$= -\sqrt{2}$

Maximum value = $c + \sqrt{a^2+b^2}$

$= \sqrt{1^2+(-1)^2}$

$= \sqrt{1+1}$

$= \sqrt{2}$

13. Find the range of the function 7Cosx – 24Sinx + 5.

Sol: Let f(x) = 7Cosx – 24Sinx + 5

Comparing with f(x) = a sinx + b cosx + c, we get a = –24, b = 7, c = 5.

Minimum value of f(x) = $c - \sqrt{a^2+b^2}$

$= 5 - \sqrt{(-24)^2+7^2}$

$= 5 - \sqrt{576+49}$

$= 5 - \sqrt{625}$

$= 5 - 25$

$= -20$

Maximum value of f(x) = $c + \sqrt{a^2 + b^2}$

$= 5 + \sqrt{625}$

$= 5 + 25$

$= 30$

$\therefore$ Range = [–20, 30]

14. If $\tan 20^0 = p$, then prove that $\dfrac{\tan 610^0 + \tan 700^0}{\tan 560^0 - \tan 470^0} = \dfrac{1 - p^2}{1 + p^2}$.

Sol: $\dfrac{\tan 610^0 + \tan 700^0}{\tan 560^0 - \tan 470^0} = \dfrac{\tan(360^0 + 250^0) + \tan(360^0 + 340^0)}{\tan(360^0 + 200^0) - \tan(360^0 + 110^0)}$

$$= \frac{\tan 250^0 + \tan 340^0}{\tan 200^0 - \tan 110^0}$$

$$= \frac{\tan(270^0 - 20^0) + \tan(360^0 - 20^0)}{\tan(180^0 + 20^0) - \tan(90^0 + 20^0)}$$

$$= \frac{\cot 20^0 - \tan 20^0}{\tan 20^0 + \text{Cot} 20^0}$$

$$= \frac{\frac{1}{p} - p}{p + \frac{1}{p}} = \frac{1 - p^2}{1 + p^2} = \text{RHS} \qquad [\because \tan 20^o = p]$$

15. Prove that $\dfrac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \dfrac{1 + \sin\theta}{\cos\theta}$.

Sol: LHS = $\dfrac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1}$

$$= \frac{\tan\theta + \sec\theta - (\sec^2\theta - \tan^2\theta)}{\tan\theta - \sec\theta + 1}$$

$$= \frac{(\tan\theta + \sec\theta) - (\sec\theta + \tan\theta)(\sec\theta - \tan\theta)}{\tan\theta - \sec\theta + 1}$$

$$= \frac{(\tan\theta + \sec\theta)(1 - \sec\theta + \tan\theta)}{(\tan\theta - \sec\theta + 1)}$$

$= \tan\theta + \text{Sec}\theta$

$= \dfrac{\sin\theta}{\cos\theta} + \dfrac{1}{\cos\theta}$

$= \dfrac{1+\sin\theta}{\cos\theta}$ = RHS

16. Prove that (1 + Cotθ – Cosecθ) (1 + tanθ + Secθ) = 2.

Sol: LHS = (1 + Cotθ – Cosecθ) (1 + tanθ + Secθ)

$$= \left(1+\frac{\cos\theta}{\sin\theta}-\frac{1}{\sin\theta}\right)\left(1+\frac{\sin\theta}{\cos\theta}+\frac{1}{\cos\theta}\right)$$

$$= \left(\frac{\sin\theta + \cos\theta - 1}{\sin\theta}\right)\left(\frac{\cos\theta+\sin\theta+1}{\cos\theta}\right)$$

$$= \frac{(\sin\theta+\cos\theta)^2 - 1}{\sin\theta\,\cos\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta - 1}{\sin\theta\,\cos\theta}$$

$$= \frac{1+2\sin\theta\cos\theta - 1}{\sin\theta\,\cos\theta}$$

$$= \frac{2\sin\theta\cos\theta}{\sin\theta\,\cos\theta}$$

= 2 = RHS

17. If θ is in 3rd Quadrant and $\tan\theta = \dfrac{\text{Cos}11^0 + \text{Sin}11^0}{\text{Cos}11^0 - \text{Sin}11^0}$, then find the value of θ.

Sol: $\tan\theta = \dfrac{\text{Cos}11^0 + \text{Sin}11^0}{\text{Cos}11^0 - \text{Sin}11^0}$

$$= \frac{\text{Cos}11^0\left(1+\dfrac{\text{Sin}11^0}{\text{Cos}11^0}\right)}{\text{Cos}11^0\left(1-\dfrac{\text{Sin}11^0}{\text{Cos}11^0}\right)}$$

$$= \frac{1+\tan11^0}{1-\tan11^0}$$

$= \tan(45^0 + 11^0)$

$= \tan56^0$

$\tan\theta = \tan(180 + 56^0) = \tan236^0$

$\Rightarrow \theta = 236^0$

18. Prove that $\frac{\text{Cos}9^0 + \text{Sin}9^0}{\text{Cos}9^0 - \text{Sin}9^0} = \text{Cot}36^0$

Sol: $\text{LHS} = \frac{\text{Cos}9^0 + \text{Sin}9^0}{\text{Cos}9^0 - \text{Sin}9^0}$

$= \frac{1 + \tan 9^0}{1 - \tan 9^0}$ [$\because$ on dividing numerator and denominator by $\cos 9^o$]

$= \tan(45^0 + 9^0)$

$= \tan 54^0$

$= \tan(90 - 36^0)$

$= \text{Cot}36^0 = \text{RHS}$

19. If $A + B = \frac{\pi}{4}$, then prove that $(1 + \tan A)(1 + \tan B) = 2$.

Sol: $A + B = 45^0$

$\Rightarrow \tan(A + B) = \tan 45^0 = 1$

$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = 1$

$\Rightarrow \tan A + \tan B = 1 - \tan A \tan B$

$\Rightarrow \tan A + \tan B + \tan A \tan B = 1$ ....(1)

$\Rightarrow$ Now $(1 + \tan A)(1 + \tan B) = 1 + \tan A + \tan B + \tan A \tan B = 2$ ($\because$ from 1)

20. Show that $\cos^2\theta + \cos^2\left(\frac{2\pi}{3} + \theta\right) + \cos^2\left(\frac{2\pi}{3} - \theta\right) = \frac{3}{2}$.

Sol: $\cos^2\left(\frac{2\pi}{3} + \theta\right) + \cos^2\left(\frac{2\pi}{3} - \theta\right)$

$= \cos^2(120 + \theta) + \cos^2(120 - \theta)$

$= (\cos 120^0 \cos\theta - \sin 120^0 \sin\theta)^2 + (\cos 120^0 \cos\theta + \sin 120^0 \sin\theta)^2$

$= 2[\cos^2 120^0 \cos^2\theta + \sin^2 120^0 \sin^2\theta]$ [$\because (a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$]

$$= 2\left[\left(\frac{-1}{2}\right)^2 \text{Cos}^2\Theta + \left(\frac{\sqrt{3}}{2}\right)^2 \text{Sin}^2\Theta\right]$$

$$= 2\left[\frac{1}{4}\text{Cos}^2\Theta + \frac{3}{4}\text{Sin}^2\Theta\right]$$

$$= \frac{2}{4}\left[\text{Cos}^2\Theta + 3\text{Sin}^2\Theta\right]$$

$= \frac{1}{2}\left[\text{Cos}^2\Theta + 3\text{Sin}^2\Theta\right]$ ....(1)

$= \text{Cos}^2\Theta + \text{Cos}^2\left(\frac{2\pi}{3}+\theta\right) + \text{Cos}^2\left(\frac{2\pi}{3}-\theta\right)$

LHS $= \text{Cos}^2\theta + \frac{1}{2}\text{Cos}^2\theta + \frac{3}{2}\text{Sin}^2\theta$ [$\because$ From (1)]

$= \frac{3}{2}\text{Cos}^2\theta + \frac{3}{2}\text{Sin}^2\theta$

$= \frac{3}{2}(\text{Cos}^2\theta + \text{Sin}^2\theta) = \frac{3}{2}$ = RHS

21. If $\frac{\text{Sin}\alpha}{a} = \frac{\text{Cos}\alpha}{b}$, then show that $a\text{Sin}2\alpha + b\text{Cos}2\alpha = b$.

Sol: $\frac{\text{Sin}\alpha}{a} = \frac{\text{Cos}\alpha}{b} = k$

$\text{Sin}\,\alpha = ak$, $\text{Cos}\,\alpha = bk$

LHS $= a\text{Sin}2\alpha + b\text{Cos}2\alpha$

$= a(2\text{Sin}\,\alpha\,\text{Cos}\,\alpha) + b(1 - 2\text{Sin}^2\alpha)$

$= a[2(ak)(bk)] + b[1-2(ak)^2]$

$= 2a^2bk^2 + b - 2a^2bk^2$

$= b$ = RHS

22. Prove that $\frac{1}{\sin 10^0} - \frac{\sqrt{3}}{\cos 10^0} = 4$.

Sol: LHS $= \frac{1}{\text{Sin}10^0} - \frac{\sqrt{3}}{\text{Cos}10^0}$

$= \frac{\text{Cos}10^0 - \sqrt{3}\text{Sin}10^0}{\text{Sin}10^0\text{Cos}10^0}$

$= \frac{2\left[\frac{1}{2}\text{Cos}10^0 - \frac{\sqrt{3}}{2}\text{Sin}10^0\right]}{\frac{1}{2}\left(2\text{Sin}10^0\text{Cos}10^0\right)}$

$= \frac{4\left[\text{Sin}30^0\text{Cos}10^0 - \text{Cos}30^0\text{Sin}10^0\right]}{\text{Sin}20^0}$

$$= \frac{4\text{Sin}(30^0 - 10^0)}{\text{Sin}20^0}$$

$$= \frac{4\text{Sin}20^0}{\text{Sin}20^0}$$

$= 4 = \text{RHS}$

23. In a ΔABC, $\tan\frac{A}{2} = \frac{5}{6}$, $\tan\frac{B}{2} = \frac{20}{37}$, then show that $\tan\frac{C}{2} = \frac{2}{5}$.

Sol: $A + B + C = 180^o \Rightarrow A + B = 180^o - C \Rightarrow \frac{A+B}{2} = 90 - \frac{C}{2}$

$$\tan\left(\frac{A+B}{2}\right) \Rightarrow \frac{A+B}{2} = 90 - \frac{C}{2}$$

$$\Rightarrow \tan\left(\frac{A}{2} + \frac{B}{2}\right) = \text{Cot}\frac{C}{2}$$

$$\Rightarrow \frac{\tan\frac{A}{2} + \tan\frac{B}{2}}{1 - \tan\frac{A}{2}\tan\frac{B}{2}} = \text{Cot}\frac{C}{2}$$

$$\Rightarrow \frac{\frac{5}{6} + \frac{20}{37}}{1 - \frac{5}{6} \cdot \frac{20}{37}} = \text{Cot}\frac{C}{2} \qquad \left[\because \tan\frac{A}{2} = \frac{5}{6},\ \tan\frac{B}{2} = \frac{20}{37}\right]$$

$$\Rightarrow \frac{\frac{185+120}{222}}{\frac{222-100}{222}} = \text{Cot}\frac{C}{2}$$

$$\Rightarrow \frac{305}{122} = \frac{1}{\tan\frac{C}{2}}$$

$$\Rightarrow \tan\frac{C}{2} = \frac{122}{305} = \frac{2}{5}$$

$$\Rightarrow \tan\frac{C}{2} = \frac{2}{5}$$

24. Prove that $\text{Cos}^2\frac{\pi}{8} + \text{Cos}^2\frac{3\pi}{8} + \text{Cos}^2\frac{5\pi}{8} + \text{Cos}^2\frac{7\pi}{8} = 2.$

Sol: LHS = $\text{Cos}^2\frac{\pi}{8} + \text{Cos}^2\frac{3\pi}{8} + \text{Cos}^2\frac{5\pi}{8} + \text{Cos}^2\frac{7\pi}{8}$

$$= \text{Cos}^2\frac{\pi}{8} + \text{Cos}^2\frac{3\pi}{8} + \text{Cos}^2\left(\pi - \frac{3\pi}{8}\right) + \text{Cos}^2\left(\pi - \frac{\pi}{8}\right)$$

$$= \text{Cos}^2\frac{\pi}{8} + \text{Cos}^2\frac{3\pi}{8} + \text{Cos}^2\frac{3\pi}{8} + \text{Cos}^2\frac{\pi}{8}$$

$$= 2\left(\text{Cos}^2\frac{\pi}{8} + \text{Cos}^2\frac{3\pi}{8}\right)$$

$$= 2\left(\text{Cos}^2\frac{\pi}{8} + \text{Cos}^2\left(\frac{\pi}{2} - \frac{\pi}{8}\right)\right)$$

$$= 2\left(\text{Cos}^2\frac{\pi}{8} + \text{Sin}^2\frac{\pi}{8}\right) \quad = 2(1) = 2 = \text{RHS}$$

25. Show that $\text{Sin}\frac{\pi}{5}.\text{Sin}\frac{2\pi}{5}.\text{Sin}\frac{3\pi}{5}.\text{Sin}\frac{4\pi}{5} = \frac{5}{16}.$

Sol: LHS = $\text{Sin}\frac{\pi}{5}.\text{Sin}\frac{2\pi}{5}.\text{Sin}\frac{3\pi}{5}.\text{Sin}\frac{4\pi}{5}$

$= \text{Sin}36^0 . \text{Sin}72^0 . \text{Sin}108^0 . \text{Sin}144^0$

$= \text{Sin}36^0 . \text{Sin}(90–18^0) . \text{Sin}(90+18^0). \text{Sin}(180–36^0)$

$= \text{Sin}36^0 . \text{Cos}18^0 . \text{Cos}18^0. \text{Sin}36^0$

$= \text{Sin}^2 36^0 . \text{Cos}^2 18^0$

$$= \left(\frac{10 - 2\sqrt{5}}{16}\right).\left(\frac{10 + 2\sqrt{5}}{16}\right)$$

$$= \frac{100 - 20}{16 \times 16} = \frac{80}{16 \times 16} = \frac{5}{16} = \text{RHS}$$

26. Prove that $\left(1 + \text{Cos}\frac{\pi}{10}\right)\left(1 + \text{Cos}\frac{3\pi}{10}\right)\left(1 + \text{Cos}\frac{7\pi}{10}\right)\left(1 + \text{Cos}\frac{9\pi}{10}\right) = \frac{1}{16}.$

Sol: LHS = $\left(1 + \text{Cos}\frac{\pi}{10}\right)\left(1 + \text{Cos}\frac{3\pi}{10}\right)\left(1 + \text{Cos}\frac{7\pi}{10}\right)\left(1 + \text{Cos}\frac{9\pi}{10}\right)$

$$= \left(1 + \text{Cos}\frac{\pi}{10}\right)\left(1 + \text{Cos}\frac{3\pi}{10}\right)\left(1 + \text{Cos}\left(\pi - \frac{3\pi}{10}\right)\right)\left(1 + \text{Cos}\left(\pi - \frac{\pi}{10}\right)\right)$$

$$= \left(1 + \cos\frac{\pi}{10}\right)\left(1 + \cos\frac{3\pi}{10}\right)\left(1 - \cos\frac{3\pi}{10}\right)\left(1 - \cos\frac{\pi}{10}\right)$$

$$= \left(1 - \cos^2\frac{\pi}{10}\right)\left(1 - \cos^2\frac{3\pi}{10}\right)$$

$$= \sin^2\frac{\pi}{10}\sin^2\frac{3\pi}{10}$$

$$= \sin^2 18^0 . \sin^2 54^0$$

$$= \left(\frac{\sqrt{5}-1}{4}\right)^2\left(\frac{\sqrt{5}+1}{4}\right)^2$$

$$= \left(\frac{(\sqrt{5}-1)(\sqrt{5}+1)}{16}\right)^2$$

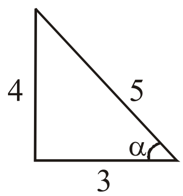
$$= \frac{(5-1)^2}{16\times16} = \frac{16}{16\times16} = \frac{1}{16} = \text{RHS}$$

27. If α, β are acute angles and $\cos\alpha = \frac{3}{5}$, $\cos\beta = \frac{5}{13}$, then show that

(i) $\sin^2\left(\frac{\alpha-\beta}{2}\right) = \frac{1}{65}$, (ii) $\cos^2\left(\frac{\alpha+\beta}{2}\right) = \frac{16}{65}$

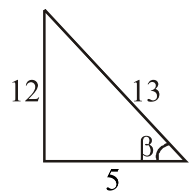
Sol: $\cos\alpha = \frac{3}{5}$

$\sin\alpha = \frac{4}{5}$



$\cos\beta = \frac{5}{13}$

$\sin\beta = \frac{12}{13}$



(i) $2\sin^2\left(\frac{\alpha-\beta}{2}\right) = 1 - \cos(\alpha-\beta)$ $\quad [\because 2\sin^2\theta = 1 - \cos2\theta]$

$= 1 - [\cos\alpha \cos\beta + \sin\alpha \sin\beta]$

$$= 1 - \left[\frac{3}{5}.\frac{5}{13} + \frac{4}{5}.\frac{12}{13}\right]$$

$$= 1 - \frac{15}{65} - \frac{48}{65}$$

$$= \frac{65-15-48}{65}$$

$$= \frac{65-63}{65} = \frac{2}{65}$$

$$\therefore \ \text{Sin}^2\left(\frac{\alpha-\beta}{2}\right) = \frac{1}{65}$$

(ii) $2\text{Cos}^2\left(\frac{\alpha+\beta}{2}\right) = 1 + \text{Cos}\,(\alpha+\beta)$ $\quad [\because 2\text{Cos}^2\theta = 1 + \text{Cos}2\theta]$

$$= 1 + \text{Cos}\alpha\ \text{Cos}\beta - \text{Sin}\alpha\ \text{Sin}\beta$$

$$= 1 + \left[\frac{3}{5}.\frac{5}{13} - \frac{4}{5}.\frac{12}{13}\right]$$

$$= 1 + \frac{15}{65} - \frac{48}{65}$$

$$= \frac{65+15-48}{65}$$

$$= \frac{80-48}{65}$$

$$\therefore \ 2\text{Cos}^2\left(\frac{\alpha+\beta}{2}\right) = \frac{32}{65}$$

$$\text{Cos}^2\left(\frac{\alpha+\beta}{2}\right) = \frac{16}{65}$$

28. A, B, C are angles in a triangle. Then prove that

Sin2A + Sin2B + Sin2C = 4 SinASinBSinC.

Sol: LHS = Sin2A + Sin2B + Sin2C

= 2Sin(A + B)Cos(A – B) + Sin2C

= 2SinC Cos(A – B) + 2SinCCosC $\quad [\because A + B + C = 180^o, A + B = 180 - C]$

= 2SinC [Cos(A–B) + CosC]

= 2SinC [Cos(A – B) – Cos (A + B)]

= 2SinC [2SinASinB]

= 4SinASinBSinC

= RHS

29. Prove that Cos2A – Cos2B + Cos2C = 1 – 4SinACosBSinC.

Sol: Cos2A – Cos2B + Cos2C

= –2Sin(A + B)Sin(A – B) + Cos2C

= –2SinCSin(A – B) + 1 – 2Sin^{2}C $\quad [\because A + B + C = 180^o, A + B = 180 - C]$

$= 1 - 2\text{Sin}C\,[\text{Sin}(A - B) + \text{Sin}C]$

$= 1 - 2\text{Sin}C[\text{Sin}(A - B) + \text{Sin}(A + B)]$

$= 1 - 2\text{Sin}C[2\text{Sin}A\text{Cos}B]$

$= 1 - 4\text{Sin}A\text{Cos}B\text{Sin}C$

30. A, B, C are angles in a triangle. Then prove that

$$\text{Sin}A + \text{Sin}B - \text{Sin}C = 4\text{Sin}\frac{A}{2}\,.\,\text{Sin}\frac{B}{2}\,.\,\text{Cos}\frac{C}{2}$$

Sol: LHS = SinA + SinB – SinC

$$= 2\text{Sin}\frac{(A+B)}{2}\text{Cos}\frac{A-B}{2} - \text{Sin}C$$

$$= 2\text{Cos}\frac{C}{2}\text{Cos}\frac{A-B}{2} - 2\text{Sin}\frac{C}{2}\text{Cos}\frac{C}{2} \qquad \left[\because \text{Sin}\left(\frac{A+B}{2}\right) = \text{Sin}\left(\frac{180-C}{2}\right) = \text{Sin}\left(90 - \frac{C}{2}\right) = \text{Cos}\frac{C}{2}\right]$$

$$= 2\text{Cos}\frac{C}{2}\left[\text{Cos}\frac{A-B}{2} - \text{Sin}\frac{C}{2}\right]$$

$$= 2\text{Cos}\frac{C}{2}\left[\text{Cos}\frac{A-B}{2} - \text{Cos}\frac{A+B}{2}\right] \qquad \left(\because \text{Sin}\frac{C}{2} = \text{Sin}\left(90 - \frac{A+B}{2}\right)\right) = \text{Cos}\left(\frac{A+B}{2}\right)$$

$$= 2\text{Cos}\frac{C}{2}\left[2\text{Sin}\frac{A}{2}\,.\,\text{Sin}\frac{B}{2}\right]$$

$$= 4\text{Sin}\frac{A}{2}\text{Sin}\frac{B}{2}\text{Cos}\frac{C}{2}$$

= RHS

31. Prove that $\text{Cos}A + \text{Cos}B - \text{Cos}C = -1 + 4\,\text{Cos}\frac{A}{2}\,\text{Cos}\frac{B}{2}\,\text{Sin}\frac{C}{2}$

Sol: LHS = CosA + CosB – CosC

$$= 2\text{Cos}\left(\frac{A+B}{2}\right)\text{Cos}\left(\frac{A-B}{2}\right) - \text{Cos}C$$

$$= 2\text{Sin}\frac{C}{2}\text{Cos}\frac{A-B}{2} - \left(1 - 2\text{Sin}^2\frac{C}{2}\right)$$

$$= 2\text{Sin}\frac{C}{2}\text{Cos}\frac{A-B}{2} - 1 + 2\text{Sin}^2\frac{C}{2}$$

$= -1 + 2\text{Sin}\frac{C}{2}\left[\text{Cos}\frac{A - B}{2} + \text{Sin}\frac{C}{2}\right]$

$= -1 + 2\text{Sin}\frac{C}{2}\left[\text{Cos}\frac{A - B}{2} + \text{Cos}\frac{A + B}{2}\right]$

$= -1 + 2\text{Sin}\frac{C}{2}\left[2\text{Cos}\frac{A}{2}.\text{Cos}\frac{B}{2}\right]$

$= -1 + 4\text{Cos}\frac{A}{2}\text{Cos}\frac{B}{2}\text{Sin}\frac{C}{2}$ = RHS

32. A, B, C are angles in a triangle. Then prove that
$\text{Sin}^2A + \text{Sin}^2B - \text{Sin}^2C = 2\text{SinA SinB CosC}$.

Sol: LHS $= \text{Sin}^2A + \text{Sin}^2B - \text{Sin}^2C$

$= 1 - \text{Cos}^2A + \text{Sin}^2B - \text{Sin}^2C$

$= 1 - (\text{Cos}^2A - \text{Sin}^2B) - \text{Sin}^2C$

$= 1 - \text{Cos}(A + B)\ \text{Cos}\ (A - B) - 1 + \text{Cos}^2C$

$= \text{CosCCos}(A - B) + \text{Cos}^2C$

$= \text{CosC}\ [\text{CosC} + \text{Cos}(A - B)]$

$= +\text{CosC}\ [\text{Cos}(A - B) - \text{Cos}(A + B)]$

$= \text{CosC}\ [2\text{SinASinB}]$

$= 2\text{SinASinBCosC}$

= RHS

33. A, B, C are angles in a triangle. Then prove that
$\text{Cos}^2A + \text{Cos}^2B - \text{Cos}^2C = 1 - 2\text{SinA SinB CosC}$

Sol: LHS $= \text{Cos}^2A + \text{Cos}^2B - \text{Cos}^2C$

$= \text{Cos}^2A + 1 - \text{Sin}^2B - \text{Cos}^2C$

$= 1 + (\text{Cos}^2A - \text{Sin}^2B) - \text{Cos}^2C$

$= 1 + \text{Cos}(A + B)\ \text{Cos}\ (A - B) - \text{Cos}^2C$

$= 1 - \text{CosC.Cos}(A - B) - \text{Cos}^2C$

$= 1 - \text{CosC}\ [\text{Cos}(A - B) + \text{CosC}]$

$= 1 - \text{CosC}\ [\text{Cos}(A - B) - \text{Cos}(A + B)]$

$= 1 - \text{CosC}\ [2\text{SinASinB}]$

$= 1 - 2\text{SinASinBCosC}$

= RHS

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Unit **9**

Hyperbolic Functions

1. $\forall\ x \in R$, $\text{Sinh}x = \dfrac{e^x - e^{-x}}{2}$

2. $\forall\ x \in R$, $\text{Cosh}x = \dfrac{e^x + e^{-x}}{2}$

3. $\forall\ x \in R$, $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

4. $\forall\ x \in R - \{0\}$, $\text{Coth}x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

5. $\forall\ x \in R$, $\text{Sech}x = \dfrac{2}{e^x + e^{-x}}$

6. $\forall\ x \in R - \{0\}$, $\text{Cosech}x = \dfrac{2}{e^x - e^{-x}}$

Note:

1) $\text{Cosh}(0) = \dfrac{e^0 + e^{-0}}{2} = \dfrac{1+1}{2} = \dfrac{2}{2} = 1$

2) $\text{Sinh}(0) = \dfrac{e^0 - e^{-0}}{2} = \dfrac{1-1}{2} = \dfrac{0}{2} = 0$

3) $\text{Cosh}(-x) = \dfrac{e^{-x} + e^{-(-x)}}{2} = \dfrac{e^{-x} + e^{x}}{2} = \text{Cosh}x$

f(-x) = f(x)

$\therefore$ Cosh$x$ is an even function.

(4) $\text{Sinh}(-x) = \dfrac{e^{-x} - e^{-(-x)}}{2} = \dfrac{e^{-x} - e^{x}}{2}$

$= -\left(\dfrac{e^x - e^{-x}}{2}\right) = -\text{Sinh}x$

$\Rightarrow f(-x) = -f(x)$ $\therefore f(x) = \text{Sinh}x$ is an odd function

IDENTITIES

1. $\forall\ x \in R$, $\text{Cosh}^2x - \text{Sinh}^2x = 1$

2. $\forall\ x \in R$, $1 - \tanh^2x = \text{Sech}^2x$

3. $\forall\ x \in R - \{0\}$, $\text{Coth}^2x - 1 = \text{Cosech}^2x$

Theorm - 1

(i) $\text{Sinh}(x + y) = \text{Sinh}x\ \text{Cosh}y + \text{Cosh}x\ \text{Sinh}y$

(ii) $\text{Sinh}(x - y) = \text{Sinh}x\ \text{Cosh}y - \text{Cosh}x\ \text{Sinh}y$

(iii) $\text{Cosh}(x + y) = \text{Cosh}x\ \text{Cosh}y + \text{Sinh}x\ \text{Sinh}y$

(iii) $\text{Cosh}(x - y) = \text{Cosh}x\ \text{Cosh}y - \text{Sinh}x\ \text{Sinh}y$

4. $\forall\ x \in R$

(i) $\text{Sinh}2x = 2\text{Sinh}x\text{Cosh}x = \dfrac{2\tanh x}{1-\tanh^2 x}$

(ii) $\text{Cosh}2x = 2\text{Cosh}^2x - 1$

$= 1 + 2\text{Sinh}^2x$

$= \dfrac{1+\tanh^2 x}{1-\tanh^2 x}$

$= \text{Cosh}^2x + \text{Sinh}^2x$

5. $\forall\ x, y \in R$

(i) $\tanh(x + y) = \dfrac{\tanh x + \tanh y}{1+\tanh x \tanh y}$

(ii) $\tanh(x - y) = \dfrac{\tanh x - \tanh y}{1-\tanh x \tanh y}$

6. $\forall\ x \in R$

(i) $\tanh 2x = \dfrac{2\tanh x}{1+\tanh^2 x}$

(ii) $\text{Coth}2x = \dfrac{\text{Coth}^2 x+1}{2\text{Coth}x}$

Theorm: $\forall\ x \in R$

$$\text{Sinh}^{-1}x = \log_e(x+\sqrt{x^2+1})$$

Theorm: $\forall\ x \in [1, \infty)$

$$\text{Cosh}^{-1}x = \log_e(x+\sqrt{x^2-1})$$

Theorm: $\forall\ x \in (-1, 1)$

$$\tanh^{-1}x = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right)$$

PROBLEMS

1. If $\text{Sinh}x = \frac{3}{4}$, then find Cosh($2x$), Sinh($2x$)

Sol: $\text{Cosh}^2x = 1 + \text{Sinh}^2x$

$$= 1 + \left(\frac{3}{4}\right)^2$$

$$= 1 + \frac{9}{16}$$

$$= \frac{25}{16}$$

$\text{Cosh}x = \frac{5}{4}$

$\text{Cosh}2x = \text{Cosh}^2x + \text{Sinh}^2x$

$$= \left(\frac{5}{4}\right)^2 + \left(\frac{3}{4}\right)^2$$

$$= \frac{25}{16} + \frac{9}{16}$$

$$= \frac{34}{16} = \frac{17}{8}$$

$\text{Sinh}2x = 2\text{Sinh}x\text{Cosh}x = 2\left(\frac{3}{4}\right)\left(\frac{5}{4}\right) = \frac{15}{8}$

2. If $\text{Sinh}x = 3$, then show that $x = \log_e(3+\sqrt{10})$

Sol: $\text{Sinh}x = 3$

$x = \text{Sinh}^{-1}(3)$

$= \log_e(3+\sqrt{3^2+1})$ $\quad [\because \text{Sinh}^{-1}x = \log_e(x+\sqrt{x^2+1})]$

$x = \log_e(3+\sqrt{10})$

3. $\forall\ n \in R$

(i) $(\text{Cosh}x - \text{Sinh}x)^n = \text{Cosh}(nx) - \text{Sinh}(nx)$

(ii) $(\text{Cosh}x + \text{Sinh}x)^n = \text{Cosh}(nx) + \text{Sinh}(nx)$

Sol: (i) $(\text{Cosh}x - \text{Sinh}x)^n = \left(\frac{e^x+e^{-x}}{2} - \frac{e^x-e^{-x}}{2}\right)^n$

$$= \left(\frac{e^x+e^{-x}-e^x+e^{-x}}{2}\right)^n$$

$$= \left(\frac{2e^{-x}}{2}\right)^n$$

$= e^{-nx}$

$$= \left(\frac{e^{nx}+e^{-nx}}{2}\right) - \left(\frac{e^{nx}-e^{-nx}}{2}\right)$$

= Cosh(nx) - Sinh(nx)

(ii) (Cosh$x$ + Sinh$x$)^n $= \left(\frac{e^{x}+e^{-x}}{2} + \frac{e^{x}-e^{-x}}{2}\right)^n$

$$= \left(\frac{e^{x}+e^{-x}+e^{x}+e^{-x}}{2}\right)^n$$

$$= \left(\frac{2e^{x}}{2}\right)^n$$

$= e^{nx}$

$$= \left(\frac{e^{nx}+e^{-nx}}{2}\right) + \left(\frac{e^{nx}-e^{-nx}}{2}\right)$$

= Cosh(nx) + Sinh(nx)

4. If $\forall\ x \in \mathrm{R}$, prove that Cosh$^4x$ – Sinh^4x = Cosh(2x).

Sol: Cosh^4x – Sinh^4x = (Cosh^2x + Sinh^2x) (Cosh^2x – Sinh^2x)

= Cosh(2x) (1)

= Cosh(2x)

5. Show that $\mathrm{Tanh}^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}\log_e 3$.

Sol: $\mathrm{Tanh}^{-1}x = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right)$

$$\mathrm{Tanh}^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}\log_e\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right)$$

$$= \frac{1}{2}\log_e\left(\frac{3/2}{1/2}\right)$$

$$= \frac{1}{2}\log_e 3$$

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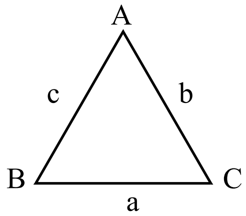
Unit **10**

PROPERTIES OF TRIANGLES

Important Points - Formulas



1. In ΔABC, BC = a, CA = b, AB = c

$a + b + c = 2S \Rightarrow S = \dfrac{a+b+c}{2}$

2. **Sine Rule:** In ΔABC

$$\frac{a}{\text{Sin}A} = \frac{b}{\text{Sin}B} = \frac{c}{\text{Sin}C} = 2R$$

R – circumradius of ΔABC.

3. **Cosine Rule:** $a^2 = b^2 + c^2 - 2bc\ \text{Cos}A$

$b^2 = c^2 + a^2 - 2ca\ \text{Cos}B$

$c^2 = a^2 + b^2 - 2ab\ \text{Cos}C$

$$\text{Cos}A = \frac{b^2 + c^2 - a^2}{2bc};\ \text{Cos}B = \frac{c^2 + a^2 - b^2}{2ca};\ \text{Cos}C = \frac{a^2 + b^2 - c^2}{2ab}$$

4. **Projection Rule:**

$a = b\text{Cos}C + c\text{Cos}B$

$b = c\text{Cos}A + a\text{Cos}C$

$c = a\text{Cos}B + b\text{Cos}A$

5. **Tangent or Napier's Rule:**

$$\tan\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{b+c}\right)\text{Cot}\frac{A}{2}$$

$$\tan\left(\frac{A-B}{2}\right) = \left(\frac{a-b}{a+b}\right)\text{Cot}\frac{C}{2}$$

$$\tan\left(\frac{C-A}{2}\right) = \left(\frac{c-a}{c+a}\right)\text{Cot}\frac{B}{2}$$

6. $\operatorname{Sin}\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$; $\operatorname{Sin}\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{ac}}$; $\operatorname{Sin}\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}$

$\operatorname{Cos}\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$; $\operatorname{Cos}\frac{B}{2}=\sqrt{\frac{s(s-b)}{ac}}$; $\operatorname{Cos}\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$

$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$; $\tan\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$; $\tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

7. ΔABC Area $\rightarrow$ $\Delta=\frac{1}{2}bc\operatorname{Sin}A=\frac{1}{2}ca\operatorname{Sin}B=\frac{1}{2}ab\operatorname{Sin}C$

$\Delta=\sqrt{s(s-a)(s-b)(s-c)}=2R^2\operatorname{Sin}A\operatorname{Sin}B\operatorname{Sin}C.$

8. $\tan\frac{A}{2}=\frac{(s-b)(s-c)}{\Delta}$; $\tan\frac{B}{2}=\frac{(s-a)(s-c)}{\Delta}$; $\tan\frac{C}{2}=\frac{(s-a)(s-b)}{\Delta}$

$\operatorname{Cot}\frac{A}{2}=\frac{s(s-a)}{\Delta}$; $\operatorname{Cot}\frac{B}{2}=\frac{s(s-b)}{\Delta}$; $\operatorname{Cot}\frac{C}{2}=\frac{s(s-c)}{\Delta}$

9. $r=\frac{\Delta}{s}$; $r_1=\frac{\Delta}{s-a}$; $r_2=\frac{\Delta}{s-b}$; $r_3=\frac{\Delta}{s-c}$

r - radius of incircle

r_1, r_2, r_3 - radii of excircles.

10. $r=\frac{\Delta}{s}=4R\operatorname{Sin}\frac{A}{2}\operatorname{Sin}\frac{B}{2}\operatorname{Sin}\frac{C}{2}$

11. $r_1=\frac{\Delta}{s-a}=4R\operatorname{Sin}\frac{A}{2}\operatorname{Cos}\frac{B}{2}\operatorname{Cos}\frac{C}{2}=S\tan\frac{A}{2}$

12. $r_2=\frac{\Delta}{s-b}=4R\operatorname{Cos}\frac{A}{2}\operatorname{Sin}\frac{B}{2}\operatorname{Cos}\frac{C}{2}=S\tan\frac{B}{2}$

13. $r_3=\frac{\Delta}{s-c}=4R\operatorname{Cos}\frac{A}{2}\operatorname{Cos}\frac{B}{2}\operatorname{Sin}\frac{C}{2}=S\tan\frac{C}{2}$

14. $\Delta^2=rr_1r_2r_3$

15. $\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}$

Short & Long Answer Questions

(Note: In all problems are refer to ΔABC)

1. In ΔABC, if a = 3, b = 4 and $\operatorname{Sin}A=\frac{3}{4}$ then find angle B.

Sol: From Sine Rule, $\frac{a}{\operatorname{Sin}A}=\frac{b}{\operatorname{Sin}B}$

aSinB = bSinA

$$\text{SinB} = \frac{\text{bSinA}}{a} = \frac{4\left(\frac{3}{4}\right)}{3} = 1 \qquad (\because \text{ from assumption } b = 4;\ a = 3;\ \text{SinA} = \frac{3}{4})$$

$\text{SinB} = 1 = \text{Sin}90^0$

$\angle B = 90^0$

2. If a = 26 cm; b = 30 cm and $\text{CosC} = \frac{63}{65}$ then find the value of c.

Sol: From Cosine rule, $c^2 = a^2 + b^2 - 2ab\text{CosC}$

$$c^2 = (26)^2 + (30)^2 - 2(26)(30)\left(\frac{63}{65}\right) \qquad (\because \text{ from assumption rule } a = 26 \text{ cm}; b = 30 \text{ cm}, \text{CosC} = \frac{63}{65})$$

$= 676 + 900 - 1512 = 64$

$c^2 = 64$

$c = 8$

3. Show that (b + c)CosA + (c + a)CosB + (a + b)CosC = a + b + c.

Sol: LHS = (b + c)CosA + (c + a)CosB + (a + b)CosC

= bCosA + cCosA + cCosB + aCosB + aCosC + bCosC

= (aCosB + bCosA) + (bCosC + cCosB) + (cCosA + aCosC)

= c + a + b ($\because$ from projection rule)

= a + b + c = RHS

$\therefore$ (b + c)CosA + (c + a)CosB + (a + b)CosC = a + b + c

4. Show that $b\text{Cos}^2\frac{C}{2} + c\text{Cos}^2\frac{B}{2} = s$.

Sol: LHS = $b\text{Cos}^2\frac{C}{2} + c\text{Cos}^2\frac{B}{2}$

$$= b\left[\frac{s(s-c)}{ab}\right] + c\left[\frac{s(s-b)}{ac}\right]$$

$$= \frac{s(s-c)}{a} + \frac{s(s-b)}{a} = \frac{s}{a}[s - c + s - b] = \frac{s}{a}[2s - b - c]$$

$$= \frac{s}{a}[a + b + c - b - c]$$

$$= \frac{s}{a}[a] = s = \text{RHS}$$

$\therefore b\text{Cos}^2\frac{C}{2} + c\text{Cos}^2\frac{B}{2} = s.$

5. Show that $\frac{a}{bc}+\frac{CosA}{a}=\frac{b}{ca}+\frac{CosB}{b}=\frac{c}{ab}+\frac{CosC}{c}$.

Sol: $\frac{a}{bc}+\frac{CosA}{a}=\frac{a}{bc}+\frac{\left(\frac{b^2+c^2-a^2}{2bc}\right)}{a}$ ($\because$ from Cosine rule, $CosA=\frac{b^2+c^2-a^2}{2bc}$)

$=\frac{a}{bc}+\frac{b^2+c^2-a^2}{2abc}=\frac{2a^2+b^2+c^2-a^2}{2abc}=\frac{a^2+b^2+c^2}{2abc}$

Similarly, $\frac{b}{ca}+\frac{CosB}{b}=\frac{a^2+b^2+c^2}{2abc}$

$\frac{c}{ab}+\frac{CosC}{c}=\frac{a^2+b^2+c^2}{2abc}$

$\therefore \frac{a}{bc}+\frac{CosA}{a}=\frac{b}{ca}+\frac{CosB}{b}=\frac{c}{ab}+\frac{CosC}{c}$

6. Show that $\frac{CosA}{a}+\frac{CosB}{b}+\frac{CosC}{c}=\frac{a^2+b^2+c^2}{2abc}$.

Sol: LHS $=\frac{CosA}{a}+\frac{CosB}{b}+\frac{CosC}{c}$

$=\frac{\left(\frac{b^2+c^2-a^2}{2bc}\right)}{a}+\frac{\left(\frac{c^2+a^2-b^2}{2ca}\right)}{b}+\frac{\left(\frac{a^2+b^2-c^2}{2ab}\right)}{c}$ ($\because$ from Cosine rule)

$=\frac{b^2+c^2-a^2}{2abc}+\frac{c^2+a^2-b^2}{2abc}+\frac{a^2+b^2-c^2}{2abc}$

$=\frac{b^2+c^2-a^2+c^2-b^2+a^2+a^2+b^2-c^2}{2abc}=\frac{a^2+b^2+c^2}{2abc}$ = RHS

$\therefore \frac{CosA}{a}+\frac{CosB}{b}+\frac{CosC}{c}=\frac{a^2+b^2+c^2}{2abc}$

7. Write the value of $aSin^2\frac{C}{2}+cSin^2\frac{A}{2}$ in terms of s, a, b, c.

Sol: $aSin^2\frac{C}{2}+cSin^2\frac{A}{2}=a\left[\frac{(s-a)(s-b)}{ab}\right]+c\left[\frac{(s-b)(s-c)}{bc}\right]$

$=\frac{(s-a)(s-b)}{b}+\frac{(s-b)(s-c)}{b}$

$= \frac{s - b}{b}[s - a + s - c] = \frac{s - b}{b}[2s - a - c] = \frac{s - b}{b}[a + b + c - a - c]$

$= \frac{s - b}{b}[b] = s - b$

$\therefore\ a\text{Sin}^2\frac{C}{2} + c\text{Sin}^2\frac{A}{2} = s - b$

8. Prove that $\text{Cot}\frac{A}{2} + \text{Cot}\frac{B}{2} + \text{Cot}\frac{C}{2} = \frac{s^2}{\Delta}$.

Sol: LHS = $\text{Cot}\frac{A}{2} + \text{Cot}\frac{B}{2} + \text{Cot}\frac{C}{2}$

$= \frac{s(s - a)}{\Delta} + \frac{s(s - b)}{\Delta} + \frac{s(s - c)}{\Delta} = \frac{s}{\Delta}[s - a + s - b + s - c]$

$= \frac{s}{\Delta}[3s - (a + b + c)] = \frac{s}{\Delta}[3s - 2s]$

$= \frac{s}{\Delta}.[s] = \frac{s^2}{\Delta} = \text{RHS}$

$\therefore\ \text{Cot}\frac{A}{2} + \text{Cot}\frac{B}{2} + \text{Cot}\frac{C}{2} = \frac{s^2}{\Delta}$

9. Prove that, $\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} = \frac{bc + ca + ab - s^2}{\Delta}$.

Sol: LHS = $\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} = \frac{(s - b)(s - c)}{\Delta} + \frac{(s - a)(s - c)}{\Delta} + \frac{(s - a)(s - b)}{\Delta}$

$= \frac{s^2 - s(b + c) + bc + s^2 - s(a + c) + ac + s^2 - s(a + b) + ab}{\Delta}$

$= \frac{3s^2 - 2s(a + b + c) + ab + bc + ca}{\Delta} = \frac{ab + bc + ca - s^2}{\Delta} = \text{RHS}$

$\therefore\ \tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} = \frac{bc + ca + ab - s^2}{\Delta}$

10. If $\text{Sin}\theta = \frac{a}{b + c}$, show that $\text{Cos}\theta = \frac{2\sqrt{bc}}{b + c}\text{Cos}\frac{A}{2}$.

Sol: Given $\text{Sin}\theta = \frac{a}{b + c}$ ....(1)

$\text{Cos}^2\theta + \text{Sin}^2\theta = 1$

$Cos^2\theta = 1 - Sin^2\theta = 1 - \left(\frac{a}{b+c}\right)^2$ [$\because$ from (1)]

$= 1 - \frac{a^2}{(b+c)^2} = \frac{(b+c)^2 - a^2}{(b+c)^2} = \frac{(b+c+a)(b+c-a)}{(b+c)^2}$

$= \frac{2s(2s-a-a)}{(b+c)^2} = \frac{2s.2(s-a)}{bc}.\frac{bc}{(b+c)^2}$

$Cos^2\theta = \frac{4s(s-a)}{bc}.\frac{bc}{(b+c)^2} = 4Cos^2\frac{A}{2}.\frac{bc}{(b+c)^2} = \frac{4bc}{(b+c)^2}.Cos^2\frac{A}{2}$

$\therefore\ Cos\theta = \frac{2\sqrt{bc}}{b+c}Cos\frac{A}{2}$

11. If $a = (b+c)Cos\theta$, show that $Sin\theta = \frac{2\sqrt{bc}}{b+c}Cos\frac{A}{2}$.

Sol: Given, $a = (b+c)Cos\theta \Rightarrow Cos\theta = \frac{a}{b+c}$

$Sin^2\theta + Cos^2\theta = 1$

$Sin^2\theta = 1 - Cos^2\theta = 1 - \left(\frac{a}{b+c}\right)^2 = 1 - \frac{a^2}{(b+c)^2} = \frac{(b+c)^2 - a^2}{(b+c)^2}$

$= \frac{(b+c+a)(b+c-a)}{(b+c)^2} = \frac{(2s)(2s-a-a)}{(b+c)^2} = \frac{(2s)2(s-a)}{(b+c)^2}$

$= 4.\frac{s(s-a)}{bc}.\frac{bc}{(b+c)^2} = 4.Cos^2\frac{A}{2}.\frac{bc}{(b+c)^2}$

$Sin^2\theta = 4\frac{bc}{(b+c)^2}.Cos^2\frac{A}{2}$

$\therefore\ Sin\theta = \frac{2\sqrt{bc}}{b+c}Cos\frac{A}{2}$

12. If $a = (b-c)\ Sec\theta$, show that $tan\theta = \frac{2\sqrt{bc}}{b-c}Sin\frac{A}{2}$.

Sol: $a = (b-c)\ Sec\theta \Rightarrow Sec\theta = \frac{a}{b-c}$

$tan^2\theta = Sec^2\theta - 1 = \left(\frac{a}{b-c}\right)^2 - 1$

$$\tan^2\theta = \frac{a^2 - (b-c)^2}{(b-c)^2} = \frac{(a+b-c)\,(a-b+c)}{(b-c)^2}$$

$$= \frac{2(s-c).\,2(s-b)}{(b-c)^2} = \frac{4(s-c)(s-b)}{bc}.\frac{bc}{(b-c)^2}$$

$$\tan^2\theta = 4\frac{bc}{(b-c)^2}\mathrm{Sin}^2\frac{A}{2}$$

$$\therefore\ \tan\theta = \frac{2\sqrt{bc}}{b-c}\mathrm{Sin}\frac{A}{2}$$

13. Prove that $\mathrm{Cot}A + \mathrm{Cot}B + \mathrm{Cot}C = \frac{a^2+b^2+c^2}{4\Delta}$.

Sol: $\mathrm{Cot}A + \mathrm{Cot}B + \mathrm{Cot}C = \sum \mathrm{Cot}A = \sum\frac{\mathrm{Cos}A}{\mathrm{Sin}A}$

$$= \sum\left(\frac{\left(\frac{b^2+c^2-a^2}{2bc}\right)}{\mathrm{Sin}A}\right) = \sum\left(\frac{b^2+c^2-a^2}{2bc\mathrm{Sin}A}\right)$$

$$= \sum\frac{b^2+c^2-a^2}{4\Delta} \qquad [\because \Delta = \frac{1}{2}bc\mathrm{Sin}A]$$

$$= \frac{b^2+c^2-a^2}{4\Delta} + \frac{c^2+a^2-b^2}{4\Delta} + \frac{a^2+b^2-c^2}{4\Delta}$$

$$= \frac{b^2+c^2-a^2+c^2+a^2-b^2+a^2+b^2-c^2}{4\Delta} = \frac{a^2+b^2+c^2}{4\Delta} = \text{RHS}$$

$$\therefore \mathrm{Cot}A + \mathrm{Cot}B + \mathrm{Cot}C = \frac{a^2+b^2+c^2}{4\Delta}$$

14. In ΔABC, if $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$, then show that $\angle C = 60^0$.

Sol: $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$

$$\frac{b+c+a+c}{(a+c)(b+c)} = \frac{3}{a+b+c}$$

$(a + b + 2c)\,(a + b + c) = 3(a + c)\,(b + c)$

$a^2 + ab + ac + ba + b^2 + bc + 2ac + 2bc + 2c^2 = 3\,[ab + ac + bc + c^2]$

$a^2 + b^2 - c^2 = ab$

$2abCosC = ab$ ($\because$ from Cosine rule)

$2CosC = 1$

$CosC = \frac{1}{2} = Cos60^0$

$\therefore \angle C = 60^0$

15. In ΔABC, if aCosA = bCosB, then show that triangle is either isosceles or right angle triangle.

Sol: aCosA = bCosB

2RSinACosA = 2RSinBCosB ($\because$ From Sine rule)

Sin2A = Sin2B = Sin(180 – 2B)

2A = 2B (or) 2A = 180 – 2B

A = B (or) A = 90 – B

A = B (or) A+ B = 90^0

$\Rightarrow$ a = b (or) $\angle C = 90^0$

$\therefore$ ΔABC is isosceles or right angle triangle.

16. If a : b : c = 7 : 8 : 9 then find CosA : CosB : CosC.

Sol: a : b : c = 7 : 8 : 9

$$\frac{a}{7} = \frac{b}{8} = \frac{c}{9} = k$$

a = 7k; b = 8k; c = 9k

$$CosA = \frac{b^2 + c^2 - a^2}{2bc} = \frac{64k^2 + 81k^2 - 49k^2}{2(8k)(9k)} = \frac{96k^2}{144k^2} = \frac{2}{3}$$

$$CosB = \frac{a^2 + c^2 - b^2}{2ac} = \frac{49k^2 + 81k^2 - 64k^2}{2(7k)(9k)} = \frac{66k^2}{126k^2} = \frac{11}{21}$$

$$CosC = \frac{a^2 + b^2 - c^2}{2ab} = \frac{49k^2 + 64k^2 - 81k^2}{2(7k)(8k)} = \frac{32k^2}{112k^2} = \frac{2}{7}$$

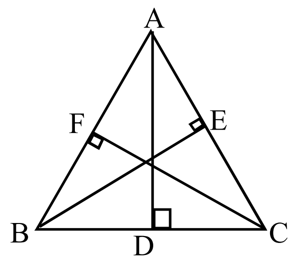
$$\therefore \ CosA : CosB : CosC = \frac{2}{3} : \frac{11}{21} : \frac{2}{7} = \left(\frac{2\times 7}{3\times 7}\right) : \frac{11}{21} : \left(\frac{2\times 3}{7\times 3}\right)$$

CosA : CosB : CosC = 14 : 11 : 6

17. In ΔABC, P_1, P_2, P_3 are altitudes, then show that $\frac{1}{P_1^2}+\frac{1}{P_2^2}+\frac{1}{P_3^2}=\frac{\text{CotA+CotB+CotC}}{\Delta}$.

Sol: In ΔABC, AD, BE, CF are altitudes.

Let $AD = P_1$, $BE = P_2$, $CF = P_3$



$$\Delta = \frac{1}{2}BC \times AD = \frac{1}{2}CA \times BE = \frac{1}{2}AB \times CF$$

$$\Delta = \frac{1}{2}a.P_1 = \frac{1}{2}b.P_2 = \frac{1}{2}c.P_3$$

$$\therefore\ P_1 = \frac{2\Delta}{a}\ ;\ P_2 = \frac{2\Delta}{b};\ P_3 = \frac{2\Delta}{c}$$

$$\frac{1}{P_1^2}+\frac{1}{P_2^2}+\frac{1}{P_3^2} = \frac{a^2}{4\Delta^2}+\frac{b^2}{4\Delta^2}+\frac{c^2}{4\Delta^2} = \frac{a^2+b^2+c^2}{4\Delta^2}$$

$$= \frac{1}{\Delta}\left(\frac{a^2+b^2+c^2}{4\Delta}\right) = \frac{1}{\Delta}(\text{CotA + CotB + CotC}) \qquad (\because \text{ From problem 13})$$

$$\therefore \frac{1}{P_1^2}+\frac{1}{P_2^2}+\frac{1}{P_3^2}=\frac{\text{CotA+CotB+CotC}}{\Delta}$$

18. Show that $\sum a\text{CotA} = 2(R + r)$.

Sol: $\text{LHS} = \sum a\text{CotA} = \sum 2R\text{SinA}\frac{\text{CosA}}{\text{SinA}} = \sum 2R\text{CosA}$

= 2R (CosA + CosB + CosC)

$$= 2R\left(1 + 4\text{Sin}\frac{A}{2}\text{Sin}\frac{B}{2}\text{Sin}\frac{C}{2}\right)$$

($\because$ from transformations $\text{CosA + CosB + CosC} = 1 + 4\text{Sin}\frac{A}{2}\text{Sin}\frac{B}{2}\text{Sin}\frac{C}{2}$)

$$= 2\left[R + 4R\text{Sin}\frac{A}{2}\text{Sin}\frac{B}{2}\text{Sin}\frac{C}{2}\right]$$

= 2[R + r]

$$\therefore\ \sum a\text{CotA} = 2(R + r)$$

19. Prove that $r(r_1 + r_2 + r_3) = ab + bc + ca - s^2$.

Sol: $\text{LHS} = r(r_1 + r_2 + r_3) = \frac{\Delta}{s}\left(\frac{\Delta}{s-a}+\frac{\Delta}{s-b}+\frac{\Delta}{s-c}\right)$

$$= \frac{\Delta^2}{s}\left(\frac{(s-b)(s-c)+(s-a)(s-c)+(s-a)(s-b)}{(s-a)(s-b)(s-c)}\right)$$

$= \frac{\Delta^2 [s^2 - s(b+c) + s^2 - s(a+c) + s^2 - s(a+b) + bc + ac + ab]}{\Delta^2}$

$= 3s^2 - 2s(a+b+c) + ab + bc + ca$

$= 3s^2 - 2s(2s) + ab + bc + ca$

$= ab + bc + ca - s^2 = RHS$

$\therefore\ r(r_1 + r_2 + r_3) = ab + bc + ca - s^2$

20. In ΔABC show that $r_1 + r_2 + r_3 - r = 4R$.

Sol: $r_1 + r_2 + r_3 - r$

$= 4R\,Sin\frac{A}{2}\,Cos\frac{B}{2}Cos\frac{C}{2} + 4R\,Cos\frac{A}{2}Sin\frac{B}{2}\,Cos\frac{C}{2} + 4R\,Cos\frac{A}{2}Cos\frac{B}{2}\,Sin\frac{C}{2}$

$- 4R\,Sin\frac{A}{2}\,Sin\frac{B}{2}\,Sin\frac{C}{2}$

$= 4RCos\frac{C}{2}\left[Sin\frac{A}{2}Cos\frac{B}{2} + Cos\frac{A}{2}Sin\frac{B}{2}\right] + 4RSin\frac{C}{2}\left[Cos\frac{A}{2}Cos\frac{B}{2} - Sin\frac{A}{2}Sin\frac{B}{2}\right]$

$= 4RCos\frac{C}{2}Sin\left(\frac{A}{2} + \frac{B}{2}\right) + 4RSin\frac{C}{2}Cos\left(\frac{A}{2} + \frac{B}{2}\right)$

$= 4RCos\frac{C}{2}Sin\left(\frac{A+B}{2}\right) + 4RSin\frac{C}{2}Cos\left(\frac{A+B}{2}\right)$

$= 4RSin\left(\frac{A+B}{2} + \frac{C}{2}\right) = 4RSin\left(\frac{A+B+C}{2}\right) = 4RSin\left(\frac{\pi}{2}\right)$

$= 4R(1) = 4R = RHS$

$\therefore\ r_1 + r_2 + r_3 - r = 4R$

21. In ΔABC prove that $r + r_1 + r_2 - r_3 = 4RCosC$.

Sol: $LHS = r + r_1 + r_2 - r_3$

$4R\,Sin\frac{A}{2}\,Sin\frac{B}{2}\,Sin\frac{C}{2} + 4R\,Sin\frac{A}{2}\,Cos\frac{B}{2}\,Cos\frac{C}{2} + 4R\,Cos\frac{A}{2}\,Sin\frac{B}{2}\,Cos\frac{C}{2}$

$- 4R\,Cos\frac{A}{2}\,Cos\frac{B}{2}\,Sin\frac{C}{2}$

$= 4RSin\frac{A}{2}\left[Sin\frac{B}{2}Sin\frac{C}{2} + Cos\frac{B}{2}Cos\frac{C}{2}\right] + 4RCos\frac{A}{2}\left[Sin\frac{B}{2}Cos\frac{C}{2} - Cos\frac{B}{2}Sin\frac{C}{2}\right]$

$= 4RSin\frac{A}{2}Cos\left(\frac{B}{2} - \frac{C}{2}\right) + 4RCos\frac{A}{2}Sin\left(\frac{B}{2} - \frac{C}{2}\right)$

$$= 4R\left[\text{Sin}\frac{A}{2}\text{Cos}\left(\frac{B-C}{2}\right)+\text{Cos}\frac{A}{2}\text{Sin}\left(\frac{B-C}{2}\right)\right]$$

$$= 4R.\text{Sin}\left(\frac{A}{2}+\frac{B-C}{2}\right) = 4R\text{Sin}\left(\frac{A+B-C}{2}\right)$$

$$= 4R\text{Sin}\left(\frac{\pi-C-C}{2}\right) = 4R\text{Sin}\left(\frac{\pi}{2}-C\right)$$

$= 4R\text{Cos}C = \text{RHS}$

$\therefore r + r_1 + r_2 - r_3 = 4R\text{Cos}C$

22. Prove that $\left(\frac{1}{r}-\frac{1}{r_1}\right)\left(\frac{1}{r}-\frac{1}{r_2}\right)\left(\frac{1}{r}-\frac{1}{r_3}\right) = \frac{abc}{\Delta^3} = \frac{4R}{r^2s^2}$.

Sol: $\left(\frac{1}{r}-\frac{1}{r_1}\right)\left(\frac{1}{r}-\frac{1}{r_2}\right)\left(\frac{1}{r}-\frac{1}{r_3}\right)$

$$= \left(\frac{s}{\Delta}-\frac{s-a}{\Delta}\right)\left(\frac{s}{\Delta}-\frac{s-b}{\Delta}\right)\left(\frac{s}{\Delta}-\frac{s-c}{\Delta}\right) = \left(\frac{s-s+a}{\Delta}\right)\left(\frac{s-s+b}{\Delta}\right)\left(\frac{s-s+c}{\Delta}\right)$$

$$= \left(\frac{a}{\Delta}\right)\left(\frac{b}{\Delta}\right)\left(\frac{c}{\Delta}\right) = \frac{abc}{\Delta^3} = \frac{4R\Delta}{\Delta^3} = \frac{4R}{\Delta^2} \qquad \left[\because \Delta = \frac{abc}{4R}, abc = 4R\Delta\right]$$

$$= \frac{4R}{(rs)^2} = \frac{4R}{r^2s^2} \qquad [\because \Delta = rs]$$

$$\left(\frac{1}{r}-\frac{1}{r_1}\right)\left(\frac{1}{r}-\frac{1}{r_2}\right)\left(\frac{1}{r}-\frac{1}{r_3}\right) = \frac{abc}{\Delta^3} = \frac{4R}{r^2s^2}$$

23. Show that $\sum\frac{r_1}{(s-b)(s-c)} = \frac{3}{r}$.

Sol: LHS $= \sum\frac{r_1}{(s-b)(s-c)} = \sum\frac{\Delta}{(s-a)(s-b)(s-c)} \qquad \left[\because r_1 = \frac{\Delta}{s-a}\right]$

$$= \sum\frac{\Delta}{\left(\frac{\Delta^2}{s}\right)} \qquad \left[\because \Delta^2 = s(s-a)(s-b)(s-c)\right]$$

$$= \sum\frac{s\Delta}{\Delta^2} = \sum\frac{s}{\Delta} = \frac{s}{\Delta}+\frac{s}{\Delta}+\frac{s}{\Delta} = \frac{3s}{\Delta}$$

$$= 3\left(\frac{s}{\Delta}\right) = 3\left(\frac{1}{r}\right) = \frac{3}{r} = \text{RHS}$$

$$\therefore \sum\frac{r_1}{(s-b)(s-c)} = \frac{3}{r}$$

24. Show that CosA + CosB + CosC = 1 + $\frac{r}{R}$.

Sol: LHS = CosA + CosB + CosC = $2\text{Cos}\left(\frac{A+B}{2}\right)\text{Cos}\left(\frac{A-B}{2}\right) + \text{CosC}$

$= 2\text{Sin}\frac{C}{2}\text{Cos}\left(\frac{A-B}{2}\right) + 1 - 2\text{Sin}^2\frac{C}{2}$ $\left(\because \frac{A+B}{2} = 90 - \frac{C}{2},\ \text{Cos}\left(\frac{A+B}{2}\right) = \sin\frac{C}{2}\right)$

$= 1 + 2\text{Sin}\frac{C}{2}\left[\text{Cos}\left(\frac{A-B}{2}\right) - \text{Sin}\frac{C}{2}\right]$

$= 1 + 2\text{Sin}\frac{C}{2}\left[\text{Cos}\left(\frac{A-B}{2}\right) - \text{Cos}\left(\frac{A+B}{2}\right)\right]$

$= 1 + 2\text{Sin}\frac{C}{2}\left[2\text{Sin}\frac{A}{2}\text{Sin}\frac{B}{2}\right]$

$= 1 + 4\text{Sin}\frac{A}{2}\text{Sin}\frac{B}{2}\text{Sin}\frac{C}{2}$

$= 1 + \frac{4R\text{Sin}\frac{A}{2}\text{Sin}\frac{B}{2}\text{Sin}\frac{C}{2}}{R} = 1 + \frac{r}{R} = \text{RHS}$

$\therefore$ CosA + CosB + CosC = 1 + $\frac{r}{R}$

25. Show that $\text{Cos}^2\frac{A}{2} + \text{Cos}^2\frac{B}{2} + \text{Cos}^2\frac{C}{2} = 2 + \frac{r}{2R}$.

Sol: $\text{Cos}^2\frac{A}{2} + \text{Cos}^2\frac{B}{2} + \text{Cos}^2\frac{C}{2} = \text{Cos}^2\frac{A}{2} + 1 - \text{Sin}^2\frac{B}{2} + \text{Cos}^2\frac{C}{2}$

$= 1 + \left(\text{Cos}^2\frac{A}{2} - \text{Sin}^2\frac{B}{2}\right) + \text{Cos}^2\frac{C}{2} = 1 + \text{Cos}\left(\frac{A+B}{2}\right)\text{Cos}\left(\frac{A-B}{2}\right) + \text{Cos}^2\frac{C}{2}$

$= 1 + \text{Sin}\frac{C}{2}\text{Cos}\left(\frac{A-B}{2}\right) + 1 - \text{Sin}^2\frac{C}{2}$ $\left[\begin{array}{l}\because \frac{A+B}{2} = 90 - \frac{C}{2} \\ \text{Cos}\left(\frac{A+B}{2}\right) = \text{Sin}\frac{C}{2}\end{array}\right]$

$= 2 + \text{Sin}\frac{C}{2}\left[\text{Cos}\left(\frac{A-B}{2}\right) - \text{Sin}\frac{C}{2}\right]$

$= 2 + \text{Sin}\frac{C}{2}\left[\text{Cos}\left(\frac{A-B}{2}\right) - \text{Cos}\left(\frac{A+B}{2}\right)\right]$

$$= 2 + \text{Sin}\frac{C}{2}\left[2\text{Sin}\frac{A}{2}\text{Sin}\frac{B}{2}\right] = 2 + 2\text{Sin}\frac{A}{2}\text{Sin}\frac{B}{2}\text{Sin}\frac{C}{2}$$

$$= 2 + \frac{(2R)\,.\,2\text{Sin}\frac{A}{2}\text{Sin}\frac{B}{2}\text{Sin}\frac{C}{2}}{2R} = 2 + \frac{4R\text{Sin}\frac{A}{2}\text{Sin}\frac{B}{2}\text{Sin}\frac{C}{2}}{2R}$$

$$= 2 + \frac{r}{2R} = \text{RHS}$$

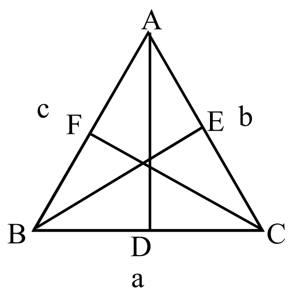
$$\therefore \text{Cos}^2\frac{A}{2} + \text{Cos}^2\frac{B}{2} + \text{Cos}^2\frac{C}{2} = 2 + \frac{r}{2R}$$

26. In ΔABC, P_1, P_2, P_3 are the altitudes drawn from the vertices A, B, C to the opposite sides respectively, then show that

(i) $\frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3} = \frac{1}{r}$ (ii) $\frac{1}{P_1} + \frac{1}{P_2} - \frac{1}{P_3} = \frac{1}{r_3}$ (iii) $P_1 P_2 P_3 = \frac{(abc)^2}{8R^3} = \frac{8\Delta^3}{abc}$

Sol: In ΔABC

$AD = P_1$, $BE = P_2$, $CF = P_3$ are altitudes.



$$\Delta = \frac{1}{2}a.P_1 = \frac{1}{2}b.P_2 = \frac{1}{2}c.P_3$$

$$2\Delta = aP_1,\ 2\Delta = bP_2;\ 2\Delta = cP_3$$

$$P_1 = \frac{2\Delta}{a};\ P_2 = \frac{2\Delta}{b};\ P_3 = \frac{2\Delta}{c}$$

(i) $$\frac{1}{P_1} + \frac{1}{P_2} + \frac{1}{P_3} = \frac{a}{2\Delta} + \frac{b}{2\Delta} + \frac{c}{2\Delta} = \frac{a+b+c}{2\Delta} = \frac{2s}{2\Delta} = \frac{s}{\Delta} = \frac{1}{r}$$

(ii) $$\frac{1}{P_1} + \frac{1}{P_2} - \frac{1}{P_3} = \frac{a}{2\Delta} + \frac{b}{2\Delta} - \frac{c}{2\Delta} = \frac{a+b-c}{2\Delta} = \frac{2s-c-c}{2\Delta} = \frac{2(s-c)}{2\Delta} = \frac{s-c}{\Delta} = \frac{1}{r_3}$$

(iii) $$P_1 P_2 P_3 = \frac{2\Delta}{a} \times \frac{2\Delta}{b} \times \frac{2\Delta}{c} = \frac{8\Delta^3}{abc}$$

$$= \frac{8\left(\frac{abc}{4R}\right)^3}{abc} = \frac{8(abc)^3}{(64R^3)abc} = \frac{(abc)^2}{8R^3}$$

$$\therefore P_1 P_2 P_3 = \frac{(abc)^2}{8R^3} = \frac{8\Delta^3}{abc}$$

27. If a = 13, b = 14, c = 15, then show that $R = \frac{65}{8}$, r = 4, $r_1 = \frac{21}{2}$, $r_2 = 12$ and $r_3 = 14$.

Sol: a = 13, b = 14, c = 15

2s = a + b + c = 13 + 14 + 15 = 42

s = 21

$\Delta^2 = s(s-a)(s-b)(s-c) = 21(21-13)(21-14)(21-15)$

$= (21)(8)(7)(6)$

$\Delta = \sqrt{21\times8\times7\times6} = \sqrt{7\times3\times2\times2\times2\times7\times2\times3} = 7\times3\times2\times2 = 84$

$\Delta = 84$

$R = \frac{abc}{4\Delta} = \frac{13\times14\times15}{4\times84} = \frac{65}{8}$

$r = \frac{\Delta}{s} = \frac{84}{21} = 4$

$r_1 = \frac{\Delta}{s-a} = \frac{84}{21-13} = \frac{84}{8} = \frac{21}{2}$

$r_2 = \frac{\Delta}{s-b} = \frac{84}{21-14} = \frac{84}{7} = 12$

$r_3 = \frac{\Delta}{s-c} = \frac{84}{21-15} = \frac{84}{6} = 14$

$\therefore R = \frac{65}{8}, r = 4, r_1 = \frac{21}{2}, r_2 = 12, r_3 = 14$

28. If $r_1 = 2, r_2 = 3, r_3 = 6$ and r = 1, then show that a = 3, b = 4, c = 5.

Sol: $r_1 = 2, r_2 = 3, r_3 = 6, r = 1$

$\Delta^2 = r\,r_1 r_2 r_3 = (1)(2)(3)(6) = 36$

$\Delta = 6$

$r = \frac{\Delta}{s} \Rightarrow s = \frac{\Delta}{r} = \frac{6}{1} = 6$

s = 6

$r_1 = \frac{\Delta}{s-a} \Rightarrow s - a = \frac{\Delta}{r_1} = \frac{6}{2} = 3$

s – a = 3

6 – a = 3

a = 3

$r_2 = \frac{\Delta}{s-b} \Rightarrow s - b = \frac{\Delta}{r_2} = \frac{6}{3} = 2$

$s - b = 2 \Rightarrow 6 - b = 2$

b = 4

$r_3 = \dfrac{\Delta}{s-c} \Rightarrow s - c = \dfrac{\Delta}{r_3} = \dfrac{6}{6} = 1$

$6 - c = 1$

$c = 5$

$\therefore$ a = 3, b = 4, c = 5

29. In ΔABC $r_1 = 8$, $r_2 = 12$, $r_3 = 24$ then find the values of a, b, c.

Sol: $r_1 = 8, r_2 = 12, r_3 = 24$

$\dfrac{1}{r} = \dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} = \dfrac{1}{8} + \dfrac{1}{12} + \dfrac{1}{24} = \dfrac{3+2+1}{24} = \dfrac{6}{24} = \dfrac{1}{4}$

$r = 4$

$\Delta^2 = r\,r_1 r_2 r_3 = (4)(8)(12)(24) = 4 \times 8 \times 12 \times 12 \times 2 = 12 \times 8 \times 12 \times 8$

$\Delta = 12 \times 8 = 96$

$\Delta = 96$

$\Delta = rs \Rightarrow s = \dfrac{\Delta}{r} = \dfrac{96}{4} = 24$

$s = 24$

$r_1 = \dfrac{\Delta}{s-a} \Rightarrow s - a = \dfrac{\Delta}{r_1} = \dfrac{96}{8} = 12$

$s - a = 12$

$24 - a = 12 \Rightarrow a = 12$

$r_2 = \dfrac{\Delta}{s-b} \Rightarrow s - b = \dfrac{\Delta}{r_2} = \dfrac{96}{12} = 8$

$s - b = 8$

$24 - b = 8$

$b = 16$

$r_3 = \dfrac{\Delta}{s-c} \Rightarrow s - c = \dfrac{\Delta}{r_3} = \dfrac{96}{24} = 4$

$s - c = 4$

$24 - c = 4$

$c = 20$

$\therefore$ a = 12, b = 16, c = 20

30. Show that $\dfrac{1}{r^2} + \dfrac{1}{r_1^2} + \dfrac{1}{r_2^2} + \dfrac{1}{r_3^2} = \dfrac{a^2+b^2+c^2}{\Delta^2}$.

Sol: $\dfrac{1}{r^2} + \dfrac{1}{r_1^2} + \dfrac{1}{r_2^2} + \dfrac{1}{r_3^2} = \dfrac{s^2}{\Delta^2} + \dfrac{(s-a)^2}{\Delta^2} + \dfrac{(s-b)^2}{\Delta^2} + \dfrac{(s-c)^2}{\Delta^2}$

$= \dfrac{1}{\Delta^2}\left[s^2 + (s-a)^2 + (s-b)^2 + (s-c)^2\right]$

$$= \frac{1}{\Delta^2}\left[s^2 + s^2 + a^2 - 2as + s^2 + b^2 - 2bs + s^2 + c^2 - 2cs\right]$$

$$= \frac{1}{\Delta^2}\left[4s^2 - 2s(a + b + c) + a^2 + b^2 + c^2\right]$$

$$= \frac{1}{\Delta^2}\left[4s^2 - 2s(2s) + a^2 + b^2 + c^2\right]$$

$$= \frac{a^2 + b^2 + c^2}{\Delta^2} = \text{RHS}$$

$$\therefore \frac{1}{r^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \frac{a^2 + b^2 + c^2}{\Delta^2}$$

31. Show that $\frac{r_1}{bc} + \frac{r_2}{ca} + \frac{r_3}{ab} = \frac{1}{r} - \frac{1}{2R}$.

Sol: LHS $= \frac{r_1}{bc} + \frac{r_2}{ca} + \frac{r_3}{ab} = \frac{1}{abc}[ar_1 + br_2 + cr_3]$

$$= \frac{1}{abc}\sum ar_1 = \frac{1}{abc}\sum 2R\operatorname{Sin}A \,.\, s\tan\frac{A}{2}$$

$$= \frac{1}{abc}\sum 2R.\, 2\operatorname{Sin}\frac{A}{2}\operatorname{Cos}\frac{A}{2}\; s\cdot\frac{\operatorname{Sin} A/2}{\operatorname{Cos} A/2} = \frac{1}{abc}.s\sum 4R\operatorname{Sin}^2\frac{A}{2}$$

$$= \frac{4Rs}{abc}\sum \operatorname{Sin}^2\frac{A}{2} = \frac{s}{\Delta}\sum\frac{1-\operatorname{Cos}A}{2} \qquad \left[\because \Delta = \frac{abc}{4R}\right]$$

$$= \frac{1}{r}\left[\frac{1 - \operatorname{Cos}A}{2} + \frac{1 - \operatorname{Cos}B}{2} + \frac{1 - \operatorname{Cos}C}{2}\right]$$

$$= \frac{1}{r}\left[\frac{1 - \operatorname{Cos}A + 1 - \operatorname{Cos}B + 1 - \operatorname{Cos}C}{2}\right] = \frac{1}{r}\left[\frac{3 - (\operatorname{Cos}A + \operatorname{Cos}B + \operatorname{Cos}C)}{2}\right]$$

$$= \frac{1}{2r}\left[3 - (\operatorname{Cos}A + \operatorname{Cos}B + \operatorname{Cos}C)\right] \qquad \left[\because \operatorname{Cos}A + \operatorname{Cos}B + \operatorname{Cos}C = 1 + 4\operatorname{Sin}\frac{A}{2}\operatorname{Sin}\frac{B}{2}\operatorname{Sin}\frac{C}{2}\right]$$

$$= \frac{1}{2r}\left[3 - \left(1 + 4\operatorname{Sin}\frac{A}{2}\operatorname{Sin}\frac{B}{2}\operatorname{Sin}\frac{C}{2}\right)\right]$$

$$= \frac{1}{2r}\left[2 - 4\operatorname{Sin}\frac{A}{2}\operatorname{Sin}\frac{B}{2}\operatorname{Sin}\frac{C}{2}\right]$$

$$= \frac{2}{2r} - \frac{4\operatorname{Sin}\frac{A}{2}\operatorname{Sin}\frac{B}{2}\operatorname{Sin}\frac{C}{2}}{2r} = \frac{1}{r} - \frac{4R\operatorname{Sin}\frac{A}{2}\operatorname{Sin}\frac{B}{2}\operatorname{Sin}\frac{C}{2}}{2rR} = \frac{1}{r} - \frac{r}{2rR}$$

$$= \frac{1}{r} - \frac{1}{2R} = \text{RHS}$$

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