# M1

1. A triangle $ABC$ with $AC = 20$ is inscribed in a circle $\omega$. A tangent $t$ to $\omega$ is drawn through $B$. The distance of $t$ from $A$ is 25 and that from $C$ is 16. If $S$ denotes the area of the triangle $ABC$, find the largest integer not exceeding $S/20$.

2. In a parallelogram $ABCD$, a point $P$ on the segment $AB$ is taken such that $\dfrac{AP}{AB} = \dfrac{61}{2022}$ and a point $Q$ on the segment $AD$ is taken such that $\dfrac{AQ}{AD} = \dfrac{61}{2065}$. If $PQ$ intersects $AC$ at $T$, find $\dfrac{AC}{AT}$ to the nearest integer.

3. In a trapezoid $ABCD$, the internal bisector of angle $A$ intersects the base $BC$ (or its extension) at the point $E$. Inscribed in the triangle $ABE$ is a circle touching the side $AB$ at $M$ and side $BE$ at the point $P$. Find the angle $DAE$ in degrees, if $AB : MP = 2$.

4. Starting with a positive integer $M$ written on the board, Alice plays the following game: in each move, if $x$ is the number on the board, she replaces it with $3x + 2$. Similarly, starting with a positive integer $N$ written on the board, Bob plays the following game: in each move, if $x$ is the number on the board, he replaces it with $2x + 27$. Given that Alice and Bob reach the same number after playing 4 moves each, find the smallest value of $M + N$.

5. Let $m$ be the smallest positive integer such that $m^2 + (m+1)^2 + \cdots + (m+10)^2$ is the square of a positive integer $n$. Find $m + n$.

6. Let $a, b$ be positive integers satisfying $a^3 - b^3 - ab = 25$. Find the largest possible value of $a^2 + b^3$.

7. Find the number of ordered pairs $(a, b)$ such that $a, b \in \{10, 11, \cdots, 29, 30\}$ and
$$\text{GCD}(a, b) + \text{LCM}(a, b) = a + b.$$

8. Suppose the prime numbers $p$ and $q$ satisfy $q^2 + 3p = 197p^2 + q$. Write $\dfrac{q}{p}$ as $l + \dfrac{m}{n}$, where $l, m, n$ are positive integers, $m < n$ and $\text{GCD}(m, n) = 1$. Find the maximum value of $l + m + n$.

9. Two sides of an integer sided triangle have lengths 18 and $x$ where $x < 100$. If there are exactly 35 possible integer values $y$ such that $18, x, y$ are the sides of a non-degenerate triangle, find the number of possible integer values $x$ can have.

10. Consider the 10-digit number $M = 9876543210$. We obtain a new 10-digit number from $M$ according to the following rule: we can choose one or more disjoint pairs of adjacent digits in $M$ and interchange the digits in these chosen pairs, keeping the remaining digits in their own places. For example, from $M = 9\underline{87}6\underline{54}3210$, by interchanging the 2 underlined pairs, and keeping the others in their places, we get $M_1 = 9\overline{78}6\overline{45}3210$. Note that any number of (disjoint) pairs can be interchanged. Find the number of new numbers that can be so obtained from $M$.

11. Let $AB$ be a diameter of a circle $\omega$ and let $C$ be a point on $\omega$, different from $A$ and $B$. The perpendicular from $C$ intersects $AB$ at $D$ and $\omega$ at $E(\neq C)$. The circle with centre at $C$ and radius $CD$ intersects $\omega$ at $P$ and $Q$. If the perimeter of the triangle $PEQ$ is 24, find the length of the side $PQ$.

12. Given $\Delta ABC$ with $\angle B = 60^\circ$ and $\angle C = 30^\circ$, let $P, Q, R$ be points on sides $BA, AC, CB$ respectively such that $BPQR$ is an isosceles trapezium with $PQ \parallel BR$ and $BP = QR$. Find the minimum possible value of $\dfrac{2[ABC]}{[BPQR]}$ where $[S]$ denotes the area of any polygon $S$.

13. Let $ABC$ be a triangle and let $D$ be a point on the segment $BC$ such that $AD = BC$. Suppose $\angle CAD = x^\circ$, $\angle ABC = y^\circ$ and $\angle ACB = z^\circ$ and $x, y, z$ are in an arithmetic progression in that order where the first term and the common difference are positive integers. Find the largest possible value of $\angle ABC$ in degrees.

14. Let $x, y, z$ be complex numbers such that

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} = 9$$
$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} = 64$$
$$\frac{x^3}{y+z} + \frac{y^3}{z+x} + \frac{z^3}{x+y} = 488$$

If $\dfrac{x}{yz} + \dfrac{y}{zx} + \dfrac{z}{xy} = \dfrac{m}{n}$ where $m, n$ are positive integers with $\text{GCD}(m, n) = 1$, find $m + n$.

15. Let $x, y$ be real numbers such that $xy = 1$. Let $T$ and $t$ be the largest and the smallest values of the expression

$$\frac{(x+y)^2 - (x-y) - 2}{(x+y)^2 + (x-y) - 2}.$$

If $T + t$ can be expressed in the form $\dfrac{m}{n}$ where $m, n$ are nonzero integers with $\text{GCD}(m, n) = 1$, find the value of $m + n$.

16. Let $a, b, c$ be reals satisfying

$$3ab + 2 = 6b, \qquad 3bc + 2 = 5c, \qquad 3ca + 2 = 4a.$$

Let $\mathbb{Q}$ denote the set of all rational numbers. Given that the product $abc$ can take two values $\dfrac{r}{s} \in \mathbb{Q}$ and $\dfrac{t}{u} \in \mathbb{Q}$, in lowest form, find $r + s + t + u$.

17. For a positive integer $n > 1$, let $g(n)$ denote the largest positive proper divisor of $n$ and $f(n) = n - g(n)$. For example, $g(10) = 5, f(10) = 5$ and $g(13) = 1, f(13) = 12$. Let $N$ be the smallest positive integer such that $f(f(f(N))) = 97$. Find the largest integer not exceeding $\sqrt{N}$.

18. Let $m, n$ be natural numbers such that

$$m + 3n - 5 = 2\,\text{LCM}(m, n) - 11\,\text{GCD}(m, n).$$

Find the maximum possible value of $m + n$.

19. Consider a string of $n$ 1's. We wish to place some $+$ signs in between so that the sum is 1000. For instance, if $n = 190$, one may put $+$ signs so as to get 11 ninety times and 1 ten times, and get the sum 1000. If $a$ is the number of positive integers $n$ for which it is possible to place $+$ signs so as to get the sum 1000, then find the sum of the digits of $a$.

20. For an integer $n \geq 3$ and a permutation $\sigma = (p_1, p_2, \ldots, p_n)$ of $\{1, 2, \ldots, n\}$, we say $p_l$ is a *landmark point* if $2 \leq l \leq n-1$ and $(p_{l-1} - p_l)(p_{l+1} - p_l) > 0$. For example, for $n = 7$, the permutation $(2, 7, 6, 4, 5, 1, 3)$ has four landmark points: $p_2 = 7, p_4 = 4, p_5 = 5$ and $p_6 = 1$. For a given $n \geq 3$, let $L(n)$ denote the number of permutations of $\{1, 2, \ldots, n\}$ with exactly one landmark point. Find the maximum $n \geq 3$ for which $L(n)$ is a perfect square.

21. An ant is at a vertex of a cube. Every 10 minutes it moves to an adjacent vertex along an edge. If $N$ is the number of one hour journeys that end at the starting vertex, find the sum of the squares of the digits of $N$.

22. A binary sequence is a sequence in which each term is equal to 0 or 1. A binary sequence is called *friendly* if each term is adjacent to at least one term that is equal to 1. For example, the sequence $0, 1, 1, 0, 0, 1, 1, 1$ is friendly. Let $F_n$ denote the number of friendly binary sequences with $n$ terms. Find the smallest positive integer $n \geq 2$ such that $F_n > 100$.

23. In a triangle $ABC$, the median $AD$ divides $\angle BAC$ in the ratio $1:2$. Extend $AD$ to $E$ such that $EB$ is perpendicular $AB$. Given that $BE = 3$, $BA = 4$, find the integer nearest to $BC^2$.

24. Let $N$ be the number of ways of distributing 52 identical balls into 4 distinguishable boxes such that no box is empty and the difference between the number of balls in any two of the boxes is not a multiple of 6. If $N = 100a + b$, where $a, b$ are positive integers less than 100, find $a + b$.

# Answers

| QNo | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Answer | 10 | 67 | 60 | 10 | 95 | 43 | 35 | 32 | 82 | 88 | 08 | 03 |

| QNo | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Answer | 59 | 16 | 25 | 18 | 19 | 70 | 09 | 03 | 74 | 11 | 29 | 81 |