LINEAR & QUADRATIC EQUATIONS

Notes / Rough Work

Linear equations

An equation where the maximum power of any variable is unity is said to be a linear equation.

e.g. $Ax + By + C = 0$

where $A \neq 0$, $B \neq 0$.

Linear equations in one variable

An equation in one variable where the maximum power of the variable is unity is said to be a linear equation in one variable. For example,

(i) $x + 1 = 3$

(ii) $y - 5 = 10$

The first equation is an example of a first degree equation or linear equation in one variable. The variable (unknown quantity) here is x. The second one is also a linear equation in variable y.

Imp.

(i) We can add the same number on both sides of an equation. e.g., $5x - 1 = 9$ is the same as $5x - 1 + 1 = 9 + 1$.

(ii) We can subtract the same number from both sides of an equation. e.g. $2a + 3 = 5$ is the same as $2a + 3 - 3 = 5 - 3$.

(iii) We can multiply both sides of an equation by the same (non-zero) number.

e.g. $\frac{3}{4}x = 6$ is the same as $4 \times \frac{3}{4}x = 6 \times 4$.

(iv) We can divide both sides of an equation by the same (non-zero) number. e.g. $2a = 8$ is the same as

$2a \div 2 = 8 \div 2$.

Solving problems with the help of linear equations

Notes / Rough Work

We can use the method of solving a linear equation to answer many difficult problems in arithmetic. The first step is to find out what facts are given in the question. Represent the unknown quantities by variables (e.g., x, a, etc.).

Then frame a linear equation using this variable and the given facts. By translating the given statement into mathematical sentences, solve the equations framed.

E1. The sum of two integers is 25. One integer is 11. Form a linear equation to find the other integer.

Sol. Sum of the two integers = 25.

One integer = 11.

Let the other integer be x.

Then the linear equation will be $11 + x = 25$.

$\therefore x = 25 - 11 = 14$.

E2. x years ago Anil's age was 10 years. What is his present age? What will be his age after 5 years?

Sol. Anil's age x years ago = 10 years.

$\therefore$ Anil's present age = $(10 + x)$ years.

His age after 5 years = $(10 + x + 5)$ years.

Once we have framed a linear equation from the given data, then we have to solve that equation and find the value of the unknown.

E3. Rahim is 3 years older than Ram. If 12 is added to Rahim's age and 3 is subtracted from Ram's age then Rahim will be 3 times as old as Ram. Find their ages.

Sol. Let Ram's age be x years.

Then Rahim's age = $(x + 3)$ years.

12 added to Rahim's age = $(x + 3 + 12)$ years

= $(x + 15)$ years .... (i)

3 subtracted from Ram's age = $(x - 3)$ years .... (ii)

It is given that (i) is 3 times (ii).

$\therefore x + 15 = 3(x - 3)$ i.e. $x + 15 = 3x - 9$

$\Rightarrow 2x = 24 \therefore x = \frac{24}{2} = 12$

Hence, Ram's age is 12 years.

Rahim's age is $(x + 3)$ years = $(12 + 3)$ years = 15 years.

E4. The sum of three consecutive integers is 24. Find the integers.

Sol. Let x be the first of the three consecutive integers.

Then the next two integers will be $x + 1$ and $x + 2$.

Sum of these three integers = 24.

i.e. $x + x + 1 + x + 2 = 24$ Or $3x + 3 = 24$

$\therefore 3x = 24 - 3 = 21 \Rightarrow x = \frac{21}{3} = 7$.

Hence, the first integer is 7.

$\therefore$ The other two integers are 7 + 1 and 7 + 2, i.e., 8 and 9. Let us verify whether our answers are correct or not.

$\Rightarrow$ 7 + 8 + 9 = 24 and 7, 8, 9 are consecutive numbers.

Notes / Rough Work

E5. A number divided by 2 is 5 less than that number. What is the number?

Sol. Let the number be x.

It is given that if the number is divided by 2 the answer will be 5 less than that number.

Hence, the linear equation is $x/2 = x - 5$.

Multiply the equation throughout by 2.

Then it becomes $x = 2x - 10$

$\therefore$ x = 10. i.e. the required number is 10.

E6. P is two years older than Q. P's father X is twice as old as P and Q is twice old as Z. The age of X and Z differs by 40 years. Find the age of X.

Sol. Let Z's age be y ....(1)

then Q's age = 2y ....(2)

and P's age = 2y + 2 ....(3)

and X's age = 4y + 4 ....(4)

It is given that difference between (4) and (1) is 40.

$\therefore 4y + 4 - y = 40.$

$\Rightarrow 3y = 40 - 4 \Rightarrow 3y = 36$

$\Rightarrow y = 12.$ $\therefore$ X's age = $4 \times 12 + 4 = 52$ Years.

E7. A man leaves Rs.8600 to be divided among 5 sons, 4 daughters and 2 nephews. If each daughter receives 4 times as much as each nephew and each son five times as much as each nephew, how much does each daughter receive?

Sol. Let each nephew receive Rs.x then each daughter receives Rs.4x

and each son receives Rs.5x

$\therefore$ Total sum will be $2 \times x + 4 \times 4x + 5 \times 5x = 8600$

$\Rightarrow 43x = 8600$ or $x = 200$.

$\therefore$ each daughter receives = $4x = 4 \times 200$ = Rs.800.

E8. Divide 86 into four parts such that the first when increased by 5, the second diminished by 1, the third multiplied by 2 and the fourth divided by 5 are equal.

Sol. Let X be the number obtained in each case. Then the four parts are

$(X - 5), (X + 1), \frac{X}{2}$ and 5X.

Now, $(X - 5) + (X + 1) + \frac{X}{2} + 5X = 86 \Rightarrow X = 12.$

Hence the required parts are 7, 13, 6 and 60.

E9. A is 29 years older than B, B is 3 years older than C and D is 2 years younger than C. Two years hence A's age will be twice the combined ages of B, C and D. Find their present ages.

Sol. Let D's age be = X years.

Then C's age = (X + 2) years,

B's age = (X + 5) years and

A's age = (X + 34) years.

Two years hence, A's, B's, C's and D's ages will be

X + 36, X + 7, X + 4 and X + 2 years respectively.

$\Rightarrow 2(X + 7 + X + 4 + X + 2) = X + 36 \Rightarrow X = 2.$

$\Rightarrow$ A's age = 36 years; B's age = 7 years, C's age = 4 years; D's age = 2 years.

Most of the questions of this topic can be solved with the help of the options provided. The equations are of the form AX + BY + C = 0
where A, B, C are real numbers.

The equation is called Linear because the graph of the equation on the X–Y Cartesian plane is a straight line.

Methods of solving linear equations

Graphical method

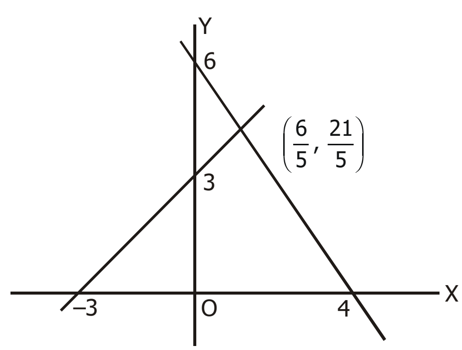
Solving equations the graphical way requires the equations to be plotted on the X–Y plane and then finding the solution.

Linear equation when plotted along the X–Y plane represents a straight line, as the degree of the equation is one. Hence, we can say that the graphical interpretation of a linear equation is a straight line plotted in the X–Y plane.

For a pair of linear equations we can say that the (X, Y) coordinates represent the point of intersection of the two straight lines.

Given the two equations $3x + 2y = 12$ and $-x + y = 3$.

Plotting these two equations graphically, we get



Thus the solution set is $\left(\frac{6}{5}, \frac{21}{5}\right)$, i.e., $x = \frac{6}{5}$ and $y = \frac{21}{5}$.

Elimination method

The principle of this method consists of multiplying the coefficients of the equations by suitable numbers such that the coefficients of at least one of the variables becomes the same in both the equations. By adding or subtracting, we get the value of one of the variables. Then by substituting this value of the variable in either of the equations, we calculate the other variable.

Notes / Rough Work

POINT TO REMEMBER

To solve a system of equations having 'n' variables, we need at least n equations.

e.g. Let the two given equations be

$3x + 2y = 12$ and $-x + y = 3$.

Multiplying the second equation by 3 and adding the two equations, we get

$-3x + 3y = 9$

$3x + 2y = 12$

$\Rightarrow 5y = 21 \Rightarrow y = \frac{21}{5}$.

Substituting this value of y in any of the equations we get $x = \frac{6}{5}$.

Substitution method

In this method one of the variables is expressed in terms of the other, thus finding out the value of the variable existing in the equation. After calculating the value of the first variable, the subsequent variables are calculated accordingly.

The two given equations are $3x + 2y = 12$ and $-x + y = 3$.

From the second equation we get, $y = 3 + x$. Substituting this value in the first equation, we get

$3x + 2(3 + x) = 12 \Rightarrow 5x = 12 - 6 \Rightarrow x = \frac{6}{5}$.

Putting this value of x in $y = 3 + x$, we get $y = \frac{21}{5}$.

E10. Solve $X + Y = 7$, $3X + 2Y = 10$ by elimination method.

Sol. We have

$X + Y = 7$ ....(1)

$3X + 2Y = 10$ ....(2)

Here, if we multiply the first equation by 2, then the coefficients of Y in both equations will be the same, viz, 2.

Hence, multiplying (1) by 2, we get

$2X + 2Y = 14$ ....(3)

Subtracting (3) from (2), we get $X = -4$.

Substituting this value of X in (1), we have $-4 + Y = 7$ or $Y = 11$.

Thus, the required values of X and Y are -4 and 11 respectively.

E11. Solve $X + Y = 3$ and $3X + 5Y = 15$ by Substitution method.

Sol. We have $X + Y = 3$ ....(1)

and $3X + 5Y = 15$ ....(2)

From (1), we have $X = 3 - Y$.

Substituting this value of X in (2), we have

$3(3 - Y) + 5Y = 15$ or $9 - 3Y + 5Y = 15$, or $9 + 2Y = 15$ or $2Y = 6$ or $Y = 3$.

Substituting this value of Y in the second equation, we have

$3X + 15 = 15$ or $3X = 0$ or $X = 0$.

Thus, the required solution is $X = 0$, $Y = 3$.

Notes / Rough Work

E12. Solve 7x + 8y = 35 and 3x + 7y = 40.

Sol. We have $7x + 8y = 35$ ....(1)

and $3x + 7y = 40$ ....(2)

From (1) we have

$7x = 35 - 8y \Rightarrow x = \frac{35-8y}{7}$.

Substituting this value of x in (2) we have $3\left(\frac{35-8y}{7}\right)+7y=40$

or $105 - 24y + 49y = 280$ or $25y = 280 - 105$

or $25y = 175$ or $y = 7$.

Substituting this value of y in (1), we have $7x + 8 \times 7 = 35$

$7x = 35 - 56$

$x = \frac{-21}{7} = -3$.

$\therefore x = -3, y = 7$.

E13. Solve for x and y. $\frac{4}{x}+\frac{10}{y}=2$ and $\frac{3}{x}+\frac{2}{y}=\frac{19}{20}$.

Sol. Let $\frac{1}{x}=p$ and $\frac{1}{y}=q$

$\therefore$ equations become

$4p + 10q = 2$ ....(1)

and $3p + 2q = \frac{19}{20}$ ....(2)

Using Elimination method, multiplying (1) by 3 and (2) by 4, we get

$12p + 30q = 6$ ....(3)

$12p + 8q = \frac{19}{5}$ ....(4)

Subtracting (4) from (3), we get

$22q = 6 - \frac{19}{5} = \frac{11}{5} \quad \therefore q = \frac{1}{10}$.

Substituting q in (1), we get

$4p + 10 \times \frac{1}{10} = 2 \therefore p = \frac{1}{4}$.

Since $\frac{1}{x}=p=\frac{1}{4} \quad \therefore x = 4$, similarly $y = 10$.

E14. Solve for x and y. $x - Py = q$ and $y - qx = P$.

Sol. We have $x - Py = q$ ....(1)

and $-qx + y = P$ ....(2)

Using Elimination method. Multiplying (1) by q,

$\therefore qx - Pqy = q^2$ ....(3)

$-qx + y = P$ ....(4)

Adding (3) and (4) we get $-Pqy + y = P + q^2 \Rightarrow y = \frac{P+q^2}{1-Pq}$.

Substituting y in (1), we get

$$x - P\left(\frac{P+q^2}{1-Pq}\right) = q \Rightarrow x - \left(\frac{P^2+Pq^2}{1-Pq}\right) = q$$

$$\Rightarrow x = q + \left(\frac{P^2+Pq^2}{1-Pq}\right) = \frac{q-Pq^2+P^2+Pq^2}{1-Pq} \Rightarrow x = \frac{P^2+q}{1-Pq}.$$

E15. Solve for x and y. $4x - 9y = 0$ and $3x + 2y = 35$.

Sol. We have $4x - 9y = 0$ ....(1)
and $3x + 2y = 35$ ....(2)
Multiplying (1) by 3 and (2) by 4, we get
$12x - 27y = 0$ ....(3)
and $12x + 8y = 140$ ....(4)
Subtracting (3) from (4), we get
$35y = 140 \therefore y = 4$
Substituting y in (1), we get
$4x - 9 \times 4 = 0 \Rightarrow 4x = 36. \therefore x = 9.$

System of equations

Two or more equations taken together form a system of equations.
e.g. $3X + 4Y = 9$; $2X + Y = 3$ form a system of equations.

Consistent system

A system (2 or 3 or more equations taken together) of two simultaneous linear equations is said to be consistent, if it has at least one solution.

Inconsistent system

A system of two simultaneous linear equations is said to be inconsistent, if it has no solutions at all.

e.g. $X + Y = 9$; $3X + 3Y = 8$

Clearly there are no values of X & Y which simultaneously satisfy the given equations. So the system is inconsistent.

E16. The equations $3x - 4y = 5$ and $12x - 16y = 20$ have

Sol. The given equations are $3x - 4y = 5$ and $3x - 4y = 5$.
Thus, there is one equation in two variables.
So, the given equations have an infinite number of solutions.

E17. The system of equations $3x + y - 1 = 0$ and $6x + 2y - 2 = 0$ have

Sol. The given equations are $3x + y = 1$ and $3x + y = 1$.
Thus, there is one equation in two variables.
So, the given equations have an infinite number of solutions.

E18. Consider the following two sets of equations
I. $2x - y = 0$ and $6x - 3y = 0$.
II. $3x - 4y = 0$ and $12x - 20y = 0$. Then,

Sol. Equations in I are $2x - y = 0$ and $2x - y = 0$.
Thus, there is one equation in two variables.
$\therefore$ given equations have an infinite number of solutions.
Equations in II are $3x - 4y = 0$ and $3x - 5y = 0$.
Solving these equations, we get $x = 0$ and $y = 0$.

Notes / Rough Work

POINT TO REMEMBER

The system $a_1X + b_1Y = c_1$
and $a_2X + b_2Y = c_2$ has

(a) a unique solution, if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$.

(b) infinitely many solutions if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

(c) no solution if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$.

The homogeneous system $a_1X + b_1Y = 0$ and $a_2X + b_2Y = 0$ has a non-zero solution only when $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ and in this case, the system has an infinite number of solutions.

E19. The equations $2x + y = 5$ and $x + 2y = 4$ are

Sol. Solving these equations, we get $x = 2$, $y = 1$.

$\therefore$ equations are consistent and have a unique solution.

E20. The equations $4x + 7y - 10 = 0$ and $20x + 35y - 50 = 0$ are

Sol. Given equations are $4x + 7y = 10$ and $4x + 7y = 10$.

Clearly, the given equations have an infinite number of solutions and are consistent also.

Applications of Equations in two variables

Step 1 Make an assumption, using two variables, say x and y.

Step 2 Construct two equations in terms of x and y.

Step 3 Solve the equations simultaneously.

E21. A number is greater than thrice the other by 2 and 6 times the smaller number exceeds the greater by 1. Find the numbers.

Sol. Let the greater number be x and the smaller number be y. Hence from the given conditions, we have

$x - 3y = 2$ ... (i)

$6y - x = 1$ ... (ii)

Adding (i) and (ii), we get $3y = 3$.

$\therefore y = 1$

Substituting $y = 1$ in (i) we have

$x - 3(1) = 2 \Rightarrow x = 2 + 3 \Rightarrow x = 5$

$\Rightarrow$ Numbers are 5 and 1.

E22. Find a fraction such that, it becomes 2/3 when 1 is added to both the numerator and denominator and becomes equal to 1/2 when 1 is subtracted from both the numerator and denominator.

Sol. Let x be the numerator and y be the denominator;

then the fraction is x/y.

According to the question,

$\frac{x+1}{y+1} = \frac{2}{3}$; that is $3x - 2y = -1$ ... (1) and

$\frac{x-1}{y-1} = \frac{1}{2}$; that is $2x - y = 1$ ... (2)

Solving (1) and (2), we get $x = 3$ and $y = 5$.

$\therefore$ The required fraction is $x/y = 3/5$.

Notes / Rough Work

E23. Few tickets of a show are sold at Rs.10 per ticket and the other tickets at Rs.8 per ticket. In all 105 tickets were sold. If the amount collected on a day was Rs.922, find the number of tickets sold at Rs.10.

Notes / Rough Work

Sol. Let 'x' tickets be sold at Rs.10 and 'y' tickets be sold at Rs.8.

$\therefore$ $x + y = 105$ and ....(i)

$10x + 8y = 922$ ....(ii)

Multiplying equation (i) by 10,

$10x + 10y = 1050$ ....(iii)

Subtracting equation (ii) from (iii),

$2y = 128$ $\therefore$ $y = 64$.

Substituting the value of 'y' in equation (i)

$x + 64 = 105$ $\therefore$ $x = 41$.

Hence, 41 tickets were sold at Rs.10.

E24. Akash has with him a total of Rs.29 in 5-rupee and 2-rupee denominations. The number of 5-rupee notes is one-half of one less than the number of 2-rupee notes. How many 5-rupee notes and 2-rupee notes does he have respectively?

Sol. Let 'x' be the number of 5-rupee notes and 'y' be the number of 2-rupee notes.

$5x + 2y = 29$ ....(i)

$x = \frac{1}{2}(y - 1)$

$2x - y = -1$ ....(ii)

Multiplying (ii) by 2,

$4x - 2y = -2$ ....(iii)

Adding (iii) and (i), $9x = 27$ $\therefore$ $x = 3$.

Substituting the value of 'x' in equation (ii),

$2 \times 3 - y = -1$ $\therefore$ $y = 7$.

E25. A fruit dealer sells 7 mangoes and buys 9 oranges, thus increases his cash by Rs.18. If he sells 8 oranges and buys 7 apples, his cash decreases by Rs.9 and if he sells 5 apples and buys 3 mangoes, increase in cash is Rs.8. What is cost of a mango, an apple and an orange?

Sol. Let cost of a mango, an apple and an orange be

x, y, z respectively. $\therefore$ As per given condition

$7x - 9z = 18$ ....(1)

$8z - 7y = -9$ ....(2)

$5y - 3x = 8$ ....(3)

Using (2) and (3) and eliminating y. Multiplying (2) by 5 and (3) by 7 we get

$40z - 35y = -45$ ....(4)

$35y - 21x = 56$ ....(5)

Adding (4) and (5) we get

$40z - 21x = 11$ ....(6)

Multiplying (1) by 3 and adding it to (6) we get

$40z - 21x + 21x - 27z = 11 + 54$

$13z = 65$ or $z = 5$.

Substituting z in (1) and (2), we get $x = 9$ and $y = 7$.

E26. The sum of the reciprocals of the ages of two brothers is five times the difference of the reciprocals of their ages. If the ratio of the product of their ages to the sum of their ages is 14.4 : 1, then find their ages.

Sol. Let the ages of the two brothers be x years and y years respectively, then, given that,

$$\frac{1}{x}+\frac{1}{y}=5\left(\frac{1}{x}-\frac{1}{y}\right) \quad ....(1)$$

and $\frac{xy}{x+y}=\frac{14.4}{1}=\frac{144}{10} \Rightarrow \frac{xy}{x+y}=\frac{72}{5}$

$$\Rightarrow \frac{1}{x}+\frac{1}{y}=\frac{5}{72} \quad ....(2)$$

From (1) and (2), $\frac{1}{x}-\frac{1}{y}=\frac{1}{72}$ ....(3)

From (2) and (3), x = 24, y = 36.

Hence, the ages of two brothers are 36 years and 24 years respectively.

Mini Revision Test # 01

DIRECTIONS: *Answer the following questions.*

1. How many solutions are possible for the system of equations
   2a – b = 6 and 6a – 3b = 12?
2. What is the value of k for which the linear equations 2x + 3y – 5 = 0,
   4x + ky – 10 = 0 have a unique solution?
3. If one number is thrice the other and their sum is 20, what are the numbers?
4. How many solutions are possible for the system of equations x + 2y = 7 and 3x + 6y = 21?
5. The sum of a natural number and its reciprocal is 37/6. What is twice of that number?

DIRECTIONS: *Fill in the blanks.*

6. The solution of the system of equations, x + y = 5 and 2x – 3y = 5 is.....
7. The solution (value of x) of the equation (2x + 3) + (5x – 4) = 13 is .....
8. The sum of three consecutive natural numbers is 153. The numbers are.....
9. A number is four times the other. If their difference is 45, the numbers are.....
10. 15 years hence a man will be just 4 times as old as he was 15 years ago. His present age is..... years.

Notes / Rough Work

Challenge Problems # 01

1. Given the quadratic equation $x^2 - (A - 3)x - (A - 2) = 0$, for what value of A will the sum of the squares of the roots be zero?
(Q. code - 110405001)

(1) -2	(2) 3
(3) 6	(4) None of these

2. Iqbal dealt some cards to Mushtaq and himself from a full pack of playing cards and laid the rest aside. Iqbal then said to Mushtaq, "If you give me a certain number of your cards, I will have four times as many cards as you have. If I give you the same number of cards, I will have thrice as many cards as you." How many cards did Iqbal have?
(Q. code - 110405002)

(1) 9	(2) 35
(3) 12	(4) 31

3. If $X^2 + Y^2 = 1$, then the value of $2(X^6 + Y^{6)} - 3(X^4 + Y^4) + 1$ is
(Q. code - 110405003)

(1) 1	(2) 0
(3) 2	(4) 4

4. The following data are available for the monsoon season of the Bengal sports club. The data is for a total of 'y' days.
   - There were races on 11 days - morning or evening.
   - Whenever there was a race in the morning, there was no race in the evening.
   - There were 8 mornings without any race.
   - There were 5 evenings without any race.

   What is the value of y? *(Q. code - 110405004)*

(1) 10	(2) 12
(3) 14	(4) Cannot be determined

5. The equation $2x - \frac{3}{x-2} = 4 - \frac{3}{x-2}$ has *(Q. code - 110405005)*

(1) No root	(2) Only one root
(3) Two equal roots	(4) None of these

Notes / Rough Work

Quadratic equations

The word QUADRATIC is derived from *'qua' means 2* and *'dratic' means order*, hence an equation in which the highest power of any variable is two is called a *quadratic equation*.

The equation is generally satisfied by two values of x but these values may be equal to each other.

General Equation: $Ax^2 + Bx + C = 0$. $(A \neq 0)$

Where A & B are coefficients of x^2 and x respectively.
e.g. $4x^2 + 9x - 5 = 0$, $6y^2 - 2y + 1 = 0$. The values of the unknown quantity (x) for which the equation is satisfied are called its roots and the process of finding the roots is called solving the equation. Geometrically a quadratic equation represents a parabola.

Solution to quadratic equations

The two most commonly used methods are

(1) Factorisation method and

(2) Quadratic equation formula.

Factorisation method

In this method, the middle term is split to make factors. This method is further clarified with the help of the following illustrations.

E27. Solve $X^2 - 6X + 8 = 0$.

Sol. We have $X^2 - 6X + 8 = 0$ or $(X - 2)(X - 4) = 0$.

$\Rightarrow$ either $(X - 2) = 0$ or $(X - 4) = 0$

$\Rightarrow X = 2$ or $X = 4$.

Quadratic equation formula

The general solution to $AX^2 + BX + C = 0$ is given as

☞ $$X = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

E28. Solve $2X^2 - 7X + 6 = 0$.

Sol. Here we have A = 2, B = –7, C = 6

$$\Rightarrow X = \frac{-(-7) \pm \sqrt{\left[(-7)^2 - 4 \times (2 \times 6)\right]}}{(2 \times 2)}$$

$$\Rightarrow X = \frac{7 \pm \sqrt{[49 - 48]}}{4} = \frac{(7 \pm 1)}{4}.$$

Now $X = (7 + 1)/4 = 8/4 = 2$ [taking + sign]

and $X = (7 - 1)/4 = 3/2$ [taking – sign]

Hence $X = 2, 3/2$.

E29. Find the roots of the equation $7p^2 - 5p - 2 = 0$.

Sol. $7p^2 - 5p - 2 = 0$. Here a = 7, b = –5, c = –2

$$\therefore\ p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \times 7 \times (-2)}}{2 \times 7}$$

$$= \frac{5 \pm \sqrt{25 + 56}}{14} = \frac{5 \pm \sqrt{81}}{14} = \frac{5 \pm 9}{14}$$

$$p = \frac{5 + 9}{14} \quad \text{or} \quad p = \frac{5 - 9}{14}$$

$$\Rightarrow p = 1 \text{ or } p = -\frac{2}{7}.$$

$\therefore$ The roots of $7p^2 - 5p - 2 = 0$ are 1 and $-\frac{2}{7}$.

Notes / Rough Work

POINT TO REMEMBER

Nature of the Roots. The roots of the quadratic equation $ax^2 + bx + c = 0$

are $\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$,

$$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

such that

$$\alpha + \beta = -\frac{b}{a} = -\frac{\text{coeff. of } x}{\text{coeff. of } x^2}$$

and $\alpha\beta = \frac{c}{a} = \frac{\text{constant term}}{\text{coeff. of } x^2}$

E30. Solve $a(x^2 + 1) = x(a^2 + 1)$.

Sol. $ax^2 + a = x(a^2 + 1)$

$ax^2 - (a^2 + 1)x + a = 0$

$$\therefore\ x = \frac{(a^2+1) \pm \sqrt{(a^2+1)^2 - 4 \times a \times a}}{2a} = \frac{(a^2+1) \pm (a^2-1)}{2a}$$

$$\therefore\ x = \frac{a^2+1+a^2-1}{2a} \text{ or } x = \frac{a^2+1-a^2+1}{2a}$$

$x = a$ or $x = \frac{1}{a}$.

E31. Solve the following quadratic equations.

(A) $x^2 - 8x + 16 = 0$.

(B) $12x^2 - 11x + 2 = 0$.

(C) $7x^2 - 13x + 3 = 0$.

(D) $15y^2 - 11y + 2 = 0$.

Sol. (A) $x^2 - 8x + 16 = 0$.

We need to find two factors whose product is 16 and sum is –8.

The factors are –4 and –4.

$\therefore\ x^2 - 4x - 4x + 16 = 0$.

$\therefore\ x(x - 4) - 4(x - 4) = 0$.

$\therefore\ (x - 4)(x - 4) = 0$.

$\therefore\ x = 4$.

(B) $12x^2 - 11x + 2 = 0$.

We need to find two factors such that their sum is –11 and product is 24.

The factors are –3 and –8.

$\therefore\ 12x^2 - 3x - 8x + 2 = 0$.

$\therefore\ 3x(4x - 1) - 2(4x - 1) = 0$.

$\therefore\ (3x - 2)(4x - 1) = 0$.

$\therefore\ 3x - 2 = 0$ or $4x - 1 = 0$.

$\therefore\ x = 2/3$ or $x = 1/4$.

(C) $7x^2 - 13x + 3 = 0$.

Using formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-(-13) \pm \sqrt{(-13)^2 - 4.3.7}}{2 \times 7} \Rightarrow x = \frac{13 \pm \sqrt{85}}{14}$$

(D) $15y^2 - 11y + 2 = 0$.

We need to find two factors such that their sum is –11 and product is 30.

The factors are –5 and –6.

$\therefore\ 15y^2 - 5y - 6y + 2 = 0$.

$\Rightarrow 5y(3y - 1) - 2(3y - 1) = 0$.

$\therefore\ (5y - 2)(3y - 1) = 0$.

$\Rightarrow\ (5y - 2) = 0$.

Or $(3y - 1) = 0$.

$\therefore\ y = 2/5$ or $y = 1/3$.

Notes / Rough Work

E32. Solve $x + y = 18$, $xy = 45$.

Sol. $x + y = 18$ .... (1)

$xy = 45$ .... (2)

Squaring (1) we get $x^2 + 2xy + y^2 = 324$.

From (2) we get $4xy = 180$.

By subtraction we get $x^2 - 2xy + y^2 = 144$ or $x - y = \pm 12$.

Combining this result with equation 1, we have,

$x + y = 18, x - y = 12 \Rightarrow x = 15, y = 3$.

$x + y = 18, x - y = -12 \Rightarrow x = 3, y = 15$.

E33. Solve $1/x - 1/y = 1/3$, $1/x^2 + 1/y^2 = 5/9$.

Sol. $1/x - 1/y = 1/3$ .... (1)

$1/x^2 + 1/y^2 = 5/9$ .... (2)

Squaring (1),

$1/x^2 - 2/xy + 1/y^2 = 1/9$ .... (3)

Subtracting (3) from (2) we get $2/xy = 4/9$.

Adding this to (2) we get

$1/x^2 + 2/xy + 1/y^2 = 1 \therefore 1/x + 1/y = \pm 1$.

Solving together with (1), $1/x = 2/3$

or $1/x = -1/3$, $1/y = 1/3$ or $1/y = -2/3$

$x = 3/2$ or -3 and $y = 3$ or $-3/2$.

Discriminant

The term $(B^2 - 4AC)$ is called the discriminant of the quadratic equation $AX^2 + BX + C$ and is denoted by D.

Imp.

- If $D > 0$, roots X_1 & X_2 are real and distinct.
- If $D = 0$, roots X_1 & X_2 are real and equal.
- If D is a perfect square, roots X_1 & X_2 are rational and distinct.
- If $D < 0$, X_1 & X_2 are imaginary, distinct and conjugates of each other.
- If X_1 and X_2 are the two roots of $aX^2 + bX + c = 0$

  then sum of roots $= X_1 + X_2 = -\frac{b}{a}$ and product of roots $= X_1 X_2 = \frac{c}{a}$.
- If roots of the quadratic equation are both positive then the signs of a and c, should be like and opposite to the sign of b.
- If roots are of opposite signs, then the signs of a and b should be like and opposite to the sign of c.
- If roots are equal in magnitude but opposite in sign, then $b = 0$.
- If roots are reciprocal of each other, then $c = a$.
- If roots of a quadratic equation are known, then the equation is

  $x^2 - x$ (Sum of the roots) + (Product of the roots) = 0.

  The Sign of the Quadratic Expression $x^2 + bx + c$.

  If the expression $ax^2 + bx + c$ is always positive, then $b^2 - 4ac$ must be negative or zero and a must be positive. If the expression $ax^2 + bx + c$ is always negative, then $b^2 - 4ac$ must be negative or zero and a must be negative.

Notes / Rough Work

POINT TO REMEMBER

The quantity $b^2 - 4ac$ is called the discriminant of the quadratic equation $ax^2 + bx + c = 0$, and it must be noted that the

(i) roots are real, irrational and distinct if $b^2 - 4ac > 0$ (not a perfect square)

(ii) roots are real and equal if $b^2 - 4ac = 0$

(iii) roots are imaginary (complex conjugate) if $b^2 - 4ac < 0$

(iv) roots are rational and unequal $b^2 - 4ac$ is a perfect square

Common Roots

Let α be a common root of the quadratic equations

$a_1 x^2 + b_1x + c_1 = 0$ and $x_2 + b_2x + c_2 = 0$

then $(c_1 a_2 - c_2 a_1)^2 = (a_1 b_2 - a_2 b_1)(b_1c_2 - b_2c_1)$.

This is the condition for a common root.

E34. Examine the nature of roots of

(i) $X^2 + 9X + 27 = 0$. (ii) $6X^2 - 13X - 5 = 0$.

(iii) $x^2 + 8x + 16 = 0$. (iv) $2x^2 - 10x + 12 = 0$.

(v) $x^2 + 11x + 32 = 0$.

Sol. (i) $D = b^2 - 4ac = (+9)^2 - 4 \times 1 \times 27$

$= 81 - 108 = -27$ which is negative.

Hence roots will be imaginary and conjugates of each other.

(ii) $D = b^2 - 4ac = (-13)^2 - 4 \times 6 \times (-5)$

$= 169 + 120 = 289$ which is a perfect square, hence roots will be real, rational and unequal.

(iii) $D = b^2 - 4ac = (8)^2 - 4 \times 16$

$= 64 - 64 = 0$.

Hence roots will be real and equal.

(iv) $D = b^2 - 4ac = (-10)^2 - 4 \times 2 \times 12$

$= 100 - 96 = 4$ which is a perfect square.

Hence roots will be real, rational and distinct.

(v) $D = b^2 - 4ac = (11)^2 - 4 \times 1 \times 32$

$= 121 - 128 = -7$ which is negative.

Hence roots will be imaginary and conjugates of each other.

Formation of equation from roots

1. If X_1 and X_2 are the two roots, then $(X - X_1)(X - X_2) = 0$ is the required equation.

2. If $(X_1 + X_2)$ and $X_1 \times X_2$ are given, then equation is $X^2 - (X_1 + X_2)X + X_1 X_2 = 0$.
   $\Rightarrow X^2 - SX + P = 0$, where S = sum of roots and P = product of roots.

Applications of Quadratic Equations

E35. The difference of a certain number and its +ve square root is 56. What is the number?

Sol. Let x^2 be the number.

$\therefore x^2 - x = 56$

$\Rightarrow x^2 - x - 56 = 0 \Rightarrow x^2 - 8x + 7x - 56 = 0$.

$\Rightarrow x(x - 8) + 7(x - 8) = 0 \Rightarrow (x - 8)(x + 7) = 0$.

$\Rightarrow$ Either $x = 8$ or $x = -7$.

As the square root is +ve $\therefore$ the number cannot be -7.

Hence answer = 64.

Notes / Rough Work

POINT TO REMEMBER

- If $b = 0$, $X_1 + X_2 = 0$ or $X_1 = -X_2$.
- If $c = 0$, one of the roots will be zero.
- If $c = a$, $X_1 = \frac{1}{X_2}$.
- If one of the roots of a quadratic equation is $K_1 + \sqrt{K_2}$, then the other root will be $K_1 - \sqrt{K_2}$ & vice versa. Thus, we see that irrational roots occur as conjugates. This is also true for imaginary roots. If one of the roots is imaginary $(m + in)$, then the other root is also imaginary $(m - in)$ & vice versa.

E36. A number of two digits is equal to three times the product of the digits and the digit in the ten's place is less by 2 than the digit in the unit's place. Find the number.

Notes / Rough Work

Sol. Let x be the digit in the ten's place and x + 2 be the digit in the unit's place.

$\therefore$ 10 x + (x + 2) = 3x (x + 2).

$\Rightarrow 10x + x + 2 = 3x^2 + 6x \Rightarrow 3x^2 - 5x - 2 = 0$

$\Rightarrow (x - 2)(3x + 1) = 0$

$\therefore x = 2 \text{ or } x = -\frac{1}{3}$.

The digit of a number cannot be $-\frac{1}{3}$. $\therefore x = 2$.

Hence the number is 24.

E37. The sum of a father's age and his son's age is 100 years. Also, one-tenth of the product of their ages, in years, exceeds the father's age by 180. How old is the son?

Sol. Let the son's age be x years and the father's age be 100 – x.

$\therefore \frac{1}{10} \times (100 - x)x = (100 - x) + 180.$

$\Rightarrow 100x - x^2 = 2800 - 10x \Rightarrow x^2 - 110x + 2800 = 0.$

$\Rightarrow (x - 70)(x - 40) = 0.$

$\therefore x = 70 \text{ or } x = 40.$

If son's age is 70, father's age will be 30. Since the son's age cannot be greater than the father's age, therefore, the son's age is 40.

E38. Find three consecutive positive integers such that the square of their sum exceeds the sum of their squares by 214.

Sol. Let the consecutive positive integers be x – 1, x and x + 1

$\therefore [x - 1 + x + x + 1]^2 = (x - 1)^2 + x^2 + (x + 1)^2 + 214$

$\Rightarrow (3x)^2 = x^2 + 1 - 2x + x^2 + x^2 + 2x + 1 + 214$

$\Rightarrow 9x^2 = 3x^2 + 216$

$\Rightarrow 6x^2 = 216 \Rightarrow x^2 = 36 \Rightarrow x = \pm 6.$

But x is a positive integer, $\therefore x = 6$.

$\therefore$ The numbers are 5, 6 and 7.

E39. Mr. Iyer distributed Rs.24 amongst his grandchildren. Had there been 4 grandchildren less, each would have got a rupee more. How many grandchildren does he have?

Sol. Let the number of grandchildren be x.

$\therefore \frac{24}{x-4} = \frac{24}{x} + 1 \Rightarrow \frac{24}{x-4} = \frac{24+x}{x}$

$\Rightarrow 24x = (24 + x)(x - 4) \Rightarrow 24x = 24x + x^2 - 4x - 96$

$\Rightarrow x^2 - 4x - 96 = 0$

$\Rightarrow (x - 12)(x + 8) = 0 \therefore x = 12 \text{ or } x = -8.$

As x cannot be negative, so x = 12.

$\therefore$ Number of grandchildren = 12.

Mini Revision Test # 02

DIRECTIONS: *Answer the following questions.*

1. Determine K so that one root of the equation $2x^2 - 5x + 2 = K$ may be zero.
2. Find m, if $x^2 + m^2 - 2mx + 3x - 5m + 3 = 0$, may have roots numerically equal but opposite in sign.
3. Find the minimum value of $x^2 - 6x + 10$ for the real values of x.
4. Find the quadratic equation, the sum and product of whose roots are 5 and 6 respectively.
5. What is the difference between the roots of $x^2 - 14x + 24 = 0$?

DIRECTIONS: *Fill in the blanks.*

6. One of the roots of $x^2 - 8x + 11 = 0$ is $4 + \sqrt{5}$. Then the other root is .... .
7. The sum of the roots of the quadratic equation $7p^2 - 35p + q - 2$ is.... .
8. The difference of a number and its reciprocal is $-15/4$. The number is .... .
9. The product of two consecutive positive numbers is 272. The smaller number is ..... .
10. The two roots of $4y^2 - py + z = 0$ are 2 and 4. The value of z is ..... .

Challenge Problems # 02

DIRECTIONS: *Answer the following questions.*

1. If p, q and r are real, the roots of $(x - p)(x - q) = r$ are always ***(Q. code - 110406001)***
   (1) Real (2) Imaginary
   (3) equal (4) Cannot be determined

2. If $a, b, c, d \in R$ then the equation $(x^2 + ax - 3b)(x^2 - cx + b)(x^2 - dx + 2b) = 0$ has ***(Q. code - 110406002)***
   (1) exactly 6 real roots (2) exactly 3 real roots
   (3) atleast 4 real roots (4) atleast 2 real roots

3. Let α, β be the roots of $x^2 - x + p = 0$ and γ, δ be the roots of $x^2 - 4x + q = 0$. If $\alpha, \beta, \gamma, \delta$ are in G.P., then the integral values of p and q respectively, are ***(Q. code - 110406003)***
   (1) $-2, -32$ (2) $-2, 3$
   (3) $-6, 3$ (4) $-6, -32$

4. If a, b are the roots of $x^2 + px + 1 = 0$, and c, d are the roots of $x^2 + qx + 1 = 0$, the value of $E = (a - c)(b - c)(a + d)(b + d)$ is ***(Q. code - 110406004)***
   (1) $p^2 - q^2$ (2) $q^2 - p^2$
   (3) $q^2 + p^2$ (4) None of these

5. If the ratio of the roots of $x^2 + ax + b = 0$ is the same as that of the roots of $x^2 + px + q = 0$, then ***(Q. code - 110406005)***
   (1) $aq^2 = pb^2$ (2) $bq^2 = pa^2$
   (3) $qb^2 = ap^2$ (4) $a^2q = bp^2$

Notes / Rough Work

Inequalities

Notes / Rough Work

An inequality states that, "one real quantity or expression is greater than or less than another real quantity or expression."

The following indicate the meaning of inequality signs.

- $a > b$ means "a is greater than b".
  ($a - b$ is a positive number).

- $a < b$ means "a is less than b".
  ($a - b$ is a negative number)

- $a \geq b$ means "a is greater than or equal to b".

- $a \leq b$ means "a is less than or equal to b".

- $0 < a < 1$ means "a is greater than zero but less than 1".

- $-2 \leq x < 2$ means "x is greater than or equal to -2 but less than 2".

Properties of Inequalities

1. For any two real numbers a and b, we have $a > b$ or $a = b$ or $a < b$.
2. If $a > b$ and $b > c$, then $a > c$.
3. If $a > b$, then $a + m > b + m$, for any real number m.
4. If $a \neq 0$, $b \neq 0$ and $a > b$, then $1/a < a/b$.
5. If $a > b$, then $am < bm$ for $m < 0$, that is, when we multiply both sides of inequality by a negative quantity, the sign of inequality is reversed.
6. If $a_1 > b_1, a_2 > b_2, \ldots, a_n > b_n$, then $a_1 + a_2 + \ldots + a_n >$ and $b_1 + b_2 + \ldots + b_n$
   $a_1 \cdot a_2 \cdot \ldots \cdot a_n > b_1 \cdot b_2 \cdot \ldots \cdot b_n$ ($a_i \geq 0$ and $b_i \geq 0$, $i = 1, 2, \ldots, n$)
7. If $x > 0$ and $a > b > 0$, then $a^x > b^x$.
8. If $a > 1$ and $x > y > 0$, then $a^x > a^y$.
9. If $0 < a < 1$ and $x > y > 0$, then $a^x < a^y$.
10. The arithmetic mean of two positive quantities is greater than or equal to their geometric mean.
    $\frac{a_1 + a_2 + \ldots + a_n}{n} \geq (a_1 a_2 \ldots a_n)^{1/n}$, that is the geometric mean of n positive quantities cannot exceed their arithmetic mean
11. The product of the factorials of two numbers whose sum is constant is least when they are equal or consecutive according as their sum is even or odd.
12. $a^2 + b^2 + c^2 \geq ab + bc + ca$.
13. $(X - \alpha)(X - \beta) > 0$ implies X does not lie between a and β.
14. $(X - \alpha)(X - \beta) < 0$ implies X lies between a and β.

The inequalities $a > b$ and $c > b$ have the same sense.
The inequalities $a > b$ and $x < y$ have opposite sense.

In brief there are three rules for producing equivalent inequalities:

(i) The same quantity can be added or subtracted to each side of an inequality.

(ii) Each side of an inequality can be multiplied or divided by the same positive quantity.

(iii) If each side of an inequality is multiplied or divided by the same negative quantity, the sign of the inequality must be reversed so that the new inequality is equivalent to the first.

Imp.

- If the signs of all the terms of an inequality are changed, then the sign of the inequality will also be reversed.
  For example, $7 > 5$ and $-7 < -5$.

- If $X > Y$, then $X^n > Y^n$ but $1/X^n < 1/Y^n$.

E40. Solve for x: If $x^2 - 3x - 40 \le 0$, $x^2 \ge 25$, $|x| > 2$.

Sol. (i) $(x - 8)(x + 5) \le 0$ Hence $-5 \le x \le 8$

(ii) $x^2 \ge 25$ Hence $x \ge 5$ or $x \le -5$

(iii) $|x| > 2$ Hence $x > 2$ or $x < -2$

(iv) The solution set for x satisfying all the above conditions is $5 \le x \le 8$ or $x = -5$.

E41. If $(x^2 + 3x - 18)(3x^2 - 7x - 6) \le 0$ solve for x.

Sol. $(x + 6)(x - 3)(3x + 2)(x - 3) \le 0 \Rightarrow (x - 3)^2 (x + 6)\left(x = \frac{2}{3}\right) \le 0$

Since, $(x - 3)^2$ is positive, $(x + 6)\left(x + \frac{2}{3}\right) \le 0$. Hence, $-6 \le x \le \frac{-2}{3}$.

E42. If $3 < \frac{3x - 4}{8} < 5$ and $x + y = 4$. Find the solution set for y.

Sol. $\frac{3x - 4}{8} > 3 \Rightarrow 3x - 4 > 24 \Rightarrow x > \frac{28}{3}; \frac{3 - 4}{8} < 5 \Rightarrow 3x < 44 \Rightarrow x < \frac{44}{3}$

and $y < 4 - \frac{28}{3}$. $\therefore$ $\frac{-32}{3} < y < \frac{-16}{3}$.

Notes / Rough Work

Notes / Rough Work

E43. If $\frac{x}{y} > 4$, x + Y < 0, |x| < 3, find the solution set for y.

Sol. Since $\frac{x}{y} > 4$, therefore both x, y have the same sign $x + y < 0 \Rightarrow x, y < 0$

$|x| < 3$. Hence $-3 < x < 0$ $\frac{x}{y} > 4$

So, $x < 4y$ (Since y is –ve) $\therefore y > \frac{x}{4}$ or $\frac{-3}{4} < y < 0$.

E44. If, x/y = y/z then which of the following is true

(1) $xz > y^2$ (2) $y^2 = xz$
(3) $xy > y^2$ (4) $xy < y^2$

Sol. Correct answer is (2)

E45. Solve for x, y if $|x| + 3y = 7$, $2x + |y - 10| = 3$.

Sol. $|x| + 3y = 7$

(a) When $x > 0$ and $y > 10$, $x + 3y = 7$ and $2x + y = 13 \Rightarrow y = 1/5$. Hence no solution possible.

(b) When $x > 0$ and $y < 10$, x 3y = 7 and $2x + 10 - y = 3 \Rightarrow y = 3$ and $x = -2$. Again, no solution possible

(c) When $x < 0$ and $y > 10$, $-2x + 6y = 14$ and $2x + y = 13 \Rightarrow y = 3\frac{6}{7}$. No solution possible.

(d) when $x < 0$ and < 10, $-2x + 6y = 14$ and $2x - y = -7 \Rightarrow y = 1.4$ an $x = -2.8$

Therefore, the unique values of x and y are –2.8 and 1.4.

E46. Solve for x : $|x + 2| > |3x - 5|$

Sol. $|x + y| > |3x - 5|$

Case 1: For $x \geq \frac{5}{3}$

$x + 2 > 3x - 5 \Rightarrow 2x < 7 \Rightarrow x < \frac{7}{2}$ i.e. $\frac{5}{3} \leq x < \frac{7}{2}$

Case 2: For $-2 \leq x \leq \frac{5}{3}$

$\Rightarrow x + 2 > 5 - 3x \Rightarrow 4x > 3 \Rightarrow x > \frac{3}{4}$ i.e. $\frac{3}{4} < x \leq \frac{5}{3}$

Case 3: For $x \leq -2$

$\Rightarrow -x - 2 > 5 - 3x \Rightarrow 2x > 7 \Rightarrow x > \frac{7}{2}$.