FUNDAMENTAL PRINCIPLE OF COUNTING AND SEQUENCING

Fundamental principles of counting

Introduction to Counting Principles

Frequently we are asked to find the number of ways a set of objects may be arranged under certain conditions.
How many ways can eight people be seated at a table?
How many different license plates are possible if each consists of three numbers and two letters?, and so on...
Finding answers to these questions and many more important and complicated ones is possible by a process called counting.
We will be discussing certain elementary aspects of the topic here, since it is so useful in mathematics and other natural and social sciences.

To determine the number of logical possibilities of some event(s) without necessarily identifying every case, two basic counting principles are used throughout. They are discussed here

The Fundamental Principle

Let us find how many numbers of two different digits can be formed from the four integers 1, 2, 3, and 4 if no digit is repeated. Any one of the four may be chosen for the tens digit of the number. With each particular choice of this type, there will remain three integers from which to choose the units digit. Therefore, for each of the four choices, there are three more choices, making a total of $4 \times 3 = 12$ numbers in all.

12, 13, 14, (1 for the tens digit),
21, 23, 24, (2 for the tens digit),
31, 32, 34, (3 for the tens digit),
41, 42, 43, (4 for the tens digit)

Notes / Rough Work

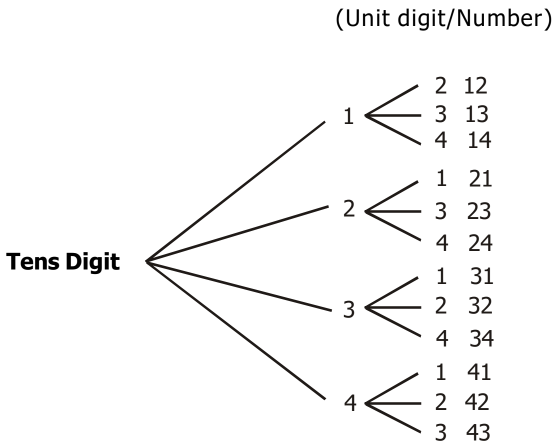
POINT TO REMEMBER

Addition Rule

If a particular job can be done in m ways, another job in n ways, and a third job in p ways, then the total number of ways any ONE job can be done is $m + n + p$.

Multiplication Rule

If a particular job can be done in m ways, another job in n ways, and a third job in p ways, then the total number of ways the three jobs can be done is given by $m \times n \times p$.



If repetitions are permitted, the result is $4 \times 4 = 16$ possible two-digit numbers instead of 12, without repetition.

The next step to this problem could be, how many numbers of three different digits could be formed with the same integers 1, 2, 3, and 4. Each of the 12 numbers listed in earlier can be regarded as representing the hundreds and tens digits. Since two of the four integers have already been selected in each case, the possibility for the new units digit must be one of the two remaining integers. For example, with the first number 41, we may have 2 or 3, forming the numbers 412 or 413. We therefore have $(4 \times 3) \times 2 = 24$ numbers, each containing three digits, which can be formed from the given integers. Similarly, if repetitions are permitted in this case the result would be $4 \times 4 \times 4 = 64$.

Sum Rule Principle

Suppose some event E can occur in m ways and a second event F can occur in n ways, and suppose both events cannot occur simultaneously, then E or F can occur in m + n ways. More generally, suppose an event E_1 can occur in n_1 ways, a second event E_2 can occur in n_2 ways, a third event E_3 can occur in n_3 ways, ..., and suppose no two of the events can occur at the same time, then one of the events (E_1, E_2 or E_3) can occur in $n_1 + n_2 + n_3 + ...$ ways.

E1 Suppose there are 8 male professors and 5 female professors teaching a calculus class. A student can choose a calculus professor in 8 + 5 = 13 ways.

Product Rule Principle

Suppose there is an event E which can occur in m ways and, independent of this event, there is a second event F which can occur in n ways. Then the total number of ways in which both E and F can occur is $m \times n$ ways.

E2 Suppose an automobile number plate contains two letters followed by three digits with the first digit not zero. How many different number plates can be printed? (assuming repetition allowed).

Each letter can be printed in 26 different ways, the first digit in 9 ways and each of the other two digits in 10 ways. Hence

$26 \times 26 \times 9 \times 10 \times 10 = 608400$ different plates can be printed.

There is a set theoretical interpretation of the given two counting principles. Specifically, suppose n(A) denotes the number of elements in a set A. Then;

(1) **Sum Rule Principle:** If A and B are disjoint sets, then

$n(A \cup B) = n(A) + n(B)$

(2) **Product Rule Principle:** Let A × B be the cartesian product of sets A and B. Then

$n(A \times B) = n(A) \times n(B)$

E3. In a Cinema hall, there are three entrance doors and two exit doors. In how many ways can a person enter the hall and then come out?

Sol. Clearly, a person can enter the hall through any of the three entrance doors. So, there are 3 ways of entering the hall. After entering the hall, the person can come out through any of the two exit doors. So, there are 2 ways of coming out. Hence the number of ways in which a person can enter the hall and then come out $= 3 \times 2 = 6$.

E4. The flag of a newly formed forum is in the form □□□, of three blocks, each to be coloured differently. If there are six different colours on the whole to choose from, how many such designs are possible?

Sol. The first block can be coloured by any one of the six given colours. So, there are 6 ways to colour the first block. Now, the second block can be coloured by any one of the remaining five colours. So, there are 5 ways to colour the second block. The third block can now be coloured by any one of the remaining four colours. So, there are 4 ways to colour the third block. $\therefore$ By the fundamental principle of counting, the number of possible designs $= 6 \times 5 \times 4 = 120$.

E5. Three persons enter a railway carriage, where there are 5 vacant seats. In how many ways can they seat themselves?

Sol. Clearly, the first person can occupy any of the 5 seats. So, there are 5 ways in which the first person can seat himself. Now, the second person can occupy any of the remaining 4 seats. So, he can be seated in 4 ways. Similarly, the third can occupy a seat in 3 ways. Hence, by the fundamental principle of counting, the required number of ways $= 5 \times 4 \times 3 = 60$.

E6. There are 4 routes between Delhi and Patna. In how many different ways can a man go from Delhi to Patna and return, if for returning.

(i) any of the routes is taken;

(ii) the same route is taken;

(iii) the same route is not taken.

Sol. (i) The man may take any route for going from Delhi to Patna. So, there are 4 ways of going from Delhi to Patna. When done so, he may return by any of the routes, i.e. in 4 different ways.

$\therefore$ Total number of ways of going to Patna and returning back to Delhi $= 4 \times 4 = 16$.

(ii) Clearly, there are 4 ways of going to Patna and in this case, there is only 1 way of returning, i.e., by the same route. $\therefore$ Total number of ways $= 4 \times 1 = 4$.

(iii) In this case, there are 4 ways of going to Patna.

But, the man can not return by the same route. So, there are 3 ways of returning back to Delhi. $\therefore$ The required number of ways $= 4 \times 3 = 12$.

E7. How many words (with or without meaning) of four distinct English alphabets are there?

Sol. We have to fill up four places by distinct letters of the alphabet. Since there are 26 letters of the alphabet, the first place can be filled by any of these letters. So, there are 26 ways of filling up the first place. Now, the second place can be filled up by any of the remaining 25 letters. So, there are 25 ways of filling up the second place. Similarly, the third and the fourth place can be filled up in 24 and 23 ways respectively. $\therefore$ The required number of words $= 26 \times 25 \times 24 \times 23 = 358800$.

Notes / Rough Work

E8. How many numbers are there between 100 and 1000 in which all the digits are distinct?

Notes / Rough Work

Sol. Clearly, we are to form all possible 3-digit numbers with distinct digits.

Note: We can not have 0 at the hundred's place. So, the hundred's place can be filled with any of the 9 digits from 1 to 9. So, there are 9 ways of filling the hundred's place.

The ten's place can now be filled with any of the remaining 8 non-zero digits or 0. So, there are 9 ways of filling the ten's place. Now, the unit's place can be filled in 8 ways, i.e., by any of the remaining 8 digits.

So the total number of required numbers = $9 \times 9 \times 8 = 648$.

E9. How many 3-digit odd numbers can be formed by using the digits 1, 2, 3, 4, 5, 6 when

(i) repetition of digits is not allowed?

(ii) repetition of digits is allowed?

Sol. For the number to be odd, we must have 1, 3 or 5 at the unit's place, i.e. there are 3 ways of filling the unit's place.

(i) Since the repetition of digits is not allowed, the ten's place can be filled with any of the remaining 5 digits.

So, this can be done in 5 ways.

For filling up the hundred's place we are left with 4 digits. Thus, the hundred's place can be filled in 4 ways.

$\therefore$ Required number of numbers = $3 \times 5 \times 4 = 60$.

(ii) Since the repetition is allowed, the ten's place can be filled up by any of the 6 digits. So, there are 6 ways of filling the ten's place. Similarly, there are 6 ways of filling the hundred's place.

$\therefore$ Required number of numbers = $3 \times 6 \times 6 = 108$.

E10. How many 9-digit numbers of different digits can be formed?

Sol. The required number of numbers are

$=(9 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2) = 3265920$.

E11. How many 3-digit numbers can be formed without using the digits 0, 2, 3, 4, 5 and 6? (repetition allowed)

Sol. Total number of required numbers $= 4 \times 4 \times 4 = 64$.

E12. How many numbers are there between 100 and 1000 such that every digit is either 2 or 9?

Sol. Clearly, repetition of digits is allowed. Each one of the unit's, ten's and hundreds place can be filled in 2 ways.

$\therefore$ Required number of numbers = $2 \times 2 \times 2 = 8$.

E13. How many odd numbers less than 1000 can be formed by using the-digits 0, 2, 5, 7 when repetition of digits is allowed?

Sol. Since each number is less than 1000, required numbers are the 1-digit, 2-digit and 3-digit numbers.

One-digit numbers: Clearly, there are two one digit odd numbers, namely 5 and 7 formed of the given digits.

Two-digit numbers: Since we are to form 2-digit odd numbers, we may put 5 or 7 at the unit's place. So, there are 2 ways of filling the unit's place.

Now, we can not use 0 at the ten's place and the repetition of digits is allowed. So, we may fill up the ten's place by any of the digits 2, 5, 7. Thus, there are 3 ways of filling the ten's place.

∴ Required number of 2-digit numbers = 2 × 3 = 6.

Three-digit numbers: To have an odd 3-digit number, we may put 5 or 7 at the unit's place. So, there are 2 ways of filling the unit's place. We may now fill up the ten's place by any of the given digits 0, 2, 5, 7. So, there are 4 ways of filling the ten's place.

We can not put 0 at the hundred's place. So, the hundred's place can be filled in 3 ways by any of the digits 2, 5, 7.

∴ Required number of 3-digit numbers = 2 × 4 × 3 = 24.

Hence, the number of required type of numbers = (2 + 6 + 24) = 32.

Notes / Rough Work

E14. How many numbers are there between 100 and 1000 such that 7 is in the unit's place?

Sol. We have to form 3-digit numbers with 7 at the unit's place. Zero can not be there at the hundred's place.

So, the hundred's place can be filled with any of the digits from 1 to 9. Thus, the hundred's place can be filled in 9 ways.

Now, the ten's place can be filled with any of the digits from 0 to 9.

So, there are 10 ways of filling the ten's place.

The unit's place can be filled in only 1 way, i.e. with 7.

∴ The number of required type of numbers = 9 × 10 × 1 = 90.

E15. How many numbers are there between 100 and 1000 such that at least one of their digits is 7?

Sol. We have to form 3-digit numbers such that at least one of their digits is 7. Now :

(i) 3-digit numbers with 7 at the unit's place:

The number of ways to fill the hundred's place = 9. [by any digit from 1 to 9]

The number of ways to fill the ten's place = 10. [by any digit from 0 to 9]

The number of ways to fill the unit's place = 1. [by 7 only]

∴ Number of such numbers = (9 × 10 × 1) = 90.

(ii) 3-digit numbers with 7 at the ten's place but not at unit's place:

The number of ways to fill the hundred's place = 9. [by any digit from 1 to 9]

The number of ways to fill the ten's place = 1. [by 7 only]

The number of ways to fill the unit's place = 9. [by any digit from 0, 1, 2, 3, 4, 5, 6, 8, 9]

∴ Number of such numbers = 9 × 1 × 9 = 81.

(iii) 3-digit numbers with 7 at the hundred's place but neither at the unit's place nor at ten's place:

The number of ways to fill the hundred's place = 1. [by 7 only]

The number of ways to fill the ten's place = 9. [by any digit from 0, 1, 2, 3, 4, 5, 6, 8, 9]

The number of ways to fill the unit's place = 9 [by any digit from 0, 1, 2, 3, 4, 5, 6, 8, 9]

∴ Number of such numbers = (1 × 9 × 9) = 81.

Hence, the number of required type of numbers = (90 + 81 + 81) = 252.

E16. How many numbers are there between 100 and 1000 which have exactly one of their digits as 7?

Notes / Rough Work

Sol. We have to form 3-digit numbers having exactly one of their digits as 7.

(i) 3-digit numbers with 7 at the unit's place but neither at the ten's place nor at the hundred's place:

The number of ways to fill the unit's place = 1. [\by 7 only]

The number of ways to fill the ten's place = 9. [by any of the digits from 0, 1, 2, 3, 4, 5, 6, 8, 9]

The number of ways to fill the hundred's place = 8 [by any of the digits from 1, 2, 3, 4, 5, 6, 8, 9]

Number of such numbers = 1 x 9 x 8 = 72.

(ii) 3-digit numbers with 7 at the ten's place but neither at the hundred's place nor at the unit's place:

The number of ways to fill the ten's place = 1. [by 7 only]

The number of ways to fill the unit's place = 9. [by any of the digits from 0, 1, 2, 3, 4, 5, 6, 8, 9]

The number of ways to fill the hundred's place = 8. [by any of the digits from 1, 2, 3, 4, 5, 6, 8, 9]

∴ Number of such numbers = 1 x 9 x 8 = 72.

(iii) 3-digit numbers with 7 at the hundred's place but neither at the unit's place nor at the ten's place:

The number of ways to fill the hundred's place = 1. [by 7 only]

The number of ways to fill the ten's place = 9. [by any of the digits from 0, 1 2, 3, 4, 5, 6, 8, 9]

The number of ways to fill the unit's place = 9. [by any of the digits from 0, 1, 2, 3, 4, 5, 6, 8, 9]

∴ Number of such numbers = $(1 \times 9 \times 9) = 81$.

Hence, the total number of required type of numbers = (72 + 72 + 81) = 225.

E17. For a set of five true or false questions, no student has written the all correct-answer and no two students have given the same sequence of answers. What is the maximum number of students in the class for this to be possible?

Sol. Clearly, there are 2 ways of answering each of the 5 questions, i.e. true or false.

∴ Total number of different sequences of answers = 2 x 2 x 2 x 2 x 2 = 32.

There is only one all-correct answer. So, the maximum number of sequences leaving all-correct answer is (32 – 1) = 31.

Since different students have given different sequences of answers, so the maximum possible number of students = 31.

FACTORIAL NOTATION

The product of the positive integers from 1 to n (inclusive) is denoted by n! (read 'n factorial')

$n! = 1 \times 2 \times 3 \ldots. (n-2) \times (n-1) \times n$

In other words, n! is defined by 1! = 1 and $n! = n.(n-1)!$

Also, we define 0! = 1.

E18.

(a) $2! = 2 \times 1 = 2$,

$3! = 3 \times 2 \times 1 = 6$,

$4! = 4 \times 3 \times 2 \times 1 = 24$

$5! = 120$, $6! = 720$.

Notes / Rough Work

(b) $\frac{8!}{6!} = \frac{8 \times 7 \times 6!}{6!} = 8 \times 7 = 56$, $12 \times 11 \times 10 = \frac{12 \times 11 \times 10 \times 9!}{9!} = \frac{12!}{9!}$

$\frac{12 \times 11 \times 10}{1 \times 2 \times 3} = 12 \times 11 \times 10 \times \frac{1}{3!} = \frac{12!}{3!9!}$

(c) $n(n-1) \ldots (n-r+1) = \frac{n(n-1)\ldots(n-r+1)(n-r)(n-r-1)\ldots 3 \times 2 \times 1}{(n-r)(n-r-1)\ldots 3 \times 2 \times 1} = \frac{n!}{(n-r)!}$

$\frac{(n-1)\ldots(n-r+1)}{1 \times 2 \times 3 \ldots (r-1)r} = n(n-1) \ldots (n+1) \cdot \frac{1}{r!} = \frac{n!}{(n-r)!} \cdot \frac{1}{r!} = \frac{n!}{r!(n-r)!}$

Binomial Coefficients

The symbol $\binom{n}{r}$ (read "^nC_r"), where r and n are positive integers with $r \le n$, is defined as follows

$$^nC_r = \frac{n(n-1)(n-2)\ldots(n-r-1)}{1 \cdot 2 \cdot 3 \ldots (r-1)r}$$

By Example (c), we see that

$$^nC_r = \frac{n(n-1)\ldots(n-r+1)}{1 \cdot 2 \cdot 3 \ldots (r-1)\, r} = \frac{n!}{r!(n-r)!}$$

But n – (n – r) = r; hence we have the following important relation:

$^nC_{n-r} = {}^nC_r$ or, in other words, if a + b = n then $^nC_a = {}^nC_b$

E19.

(a) $^8C_2 = \frac{8 \times 7}{1 \times 2} = 28$, $^9C_4 = \frac{9 \times 8 \times 7 \times 6}{1 \times 2 \times 3 \times 4} = 126$, $^{12}C_5 = \frac{12 \times 11 \times 10 \times 9 \times 8}{1 \times 2 \times 3 \times 4 \times 5} = 792$

$^{10}C_3 = \frac{10 \times 9 \times 8}{1 \times 2 \times 3} = 120$, $^{13}C_1 = \frac{13}{1} = 13$.

Note that ^nC_r has exactly r factors in both the numerator and the denominator.

(b) Compute $^{10}C_7$. By definition, $^{10}C_7 = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7} = 120$

On the other hand, 10 – 7 = 3 and so we can also compute $^{10}C_7$ as follows

$^{10}C_7 = \frac{10 \times 9 \times 8}{1 \times 2 \times 3} = 120$

Observe that the second method saves space and time.

The Pigeonhole Principle

A lot of simple results in at times come from the following almost obvious statement.

If n pigeonholes are occupied by n + 1 or more pigeons, then at least one pigeonhole is occupied by more than one pigeon. This principle can be applied to many problems where we want to show that a given situation can occur.

Notes / Rough Work

E20.

(a) Suppose a department contains 13 professors. Then atleast two of the professors (pigeons) were born in the same month (pigeonholes).

(b) Suppose a laundry bag contains many red, white, and blue socks. Then one need only grab four socks (pigeons) to be sure of getting a pair with the same colour (pigeonholes).

Generalized Pigeonhole Principle

If n pigeonholes are occupied by kn + 1 or more pigeons, where k is a positive integer, then at least one pigeonhole is occupied by k + 1 or more pigeons.

E21.

(a) Find the minimum number of students in a class to be sure that three of them are born in the same month. Here the n = 12 months are the pigeonholes and k + 1 = 3, or k = 2. Hence among any kn + 1 = 25 students (pigeons), three of them are born in the same month.

(b) Suppose a laundrybag contains many red, white, and blue socks. Find the minimum number of socks that one needs to choose in order to get two pairs (four socks) of the same colour.

Here there are n = 3 colours (pigeonholes) and k + 1 = 4, or k = 3. Thus among any kn + 1 = 10 socks (pigeons), four of them have the same colour.

FACTORIAL NOTATION AND BINOMIAL COEFFICIENTS

E22. Compute 4!, 5!, 6! and 7!

Sol. $4! = 1 \times 2 \times 3 \times 4 = 24!$

$5! = 1 \times 2 \times 3 \times 4 \times 5 = 5 \times (4!) = 5 \times (24) = 120$

$6! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 = 6 \times (5!) = 6 \times (120) = 720$

$7! = 7 \times (6!) = 7 \times (720) = 5040$

E23. Compute (a) $\frac{13!}{11!}$; (b) $\frac{7!}{10!}$

Sol. (a) $\frac{13!}{11!} = \frac{13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 2 \times 1}{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = 13 \times 12 = 156.$

Alternatively, this could be solved as follows:

$$\frac{13!}{11!} = \frac{13 \times 12 \times 11!}{11!} = 13 \times 12 = 156.$$

(b) $\frac{7!}{10!} = \frac{7!}{10 \times 9 \times 8 \times 7!} = \frac{1}{10 \times 9 \times 8} = \frac{1}{720}.$

E24. Simplify (a) $\frac{n!}{(n-1)!}$; (b) $\frac{n+2!}{n!}$

Sol. (a) $\frac{n!}{(n-1)!} = \frac{n(n-1)(n-2)\ldots3.2.1}{(n-1)(n-2)\ldots3.2.1} = n$; alternatively, $\frac{n!}{(n-1)!} = \frac{n(n-1)!}{(n-1)!} = n$

(b) $\frac{(n+2)!}{n!} = \frac{(n+2)(n+1)n!}{n!} = (n + 2)(n + 1) = n^2 + 3n + 2.$

Notes / Rough Work

E25. Compute: (a) $\binom{16}{3}$; (b) $\binom{12}{4}$

Recall that there are as many factors in the numerator as in the denominator.

Sol. (a) $^{16}C_3 = \dfrac{16 \times 15 \times 14}{1 \times 2 \times 3} = 560$

(b) $^{12}C_4 = \dfrac{12 \times 11 \times 10 \times 9}{1 \times 2 \times 3 \times 4} = 495$

E26. Compute: (a) $^8C_5$ (b) $^9C_7$

Sol. (a) $^8C_5 = \dfrac{8 \times 7 \times 6 \times 5 \times 4}{1 \times 2 \times 3 \times 4 \times 5} = 56$ or, since 8 – 5 = 3, $^8C_5 = {}^8C_5 = \dfrac{8 \times 7. \times 6}{1 \times 2 \times 3} = 56$

(b) Since 9 – 7 = 2, we have $^9C_7 = {}^9C_2 = \dfrac{9 \times 8}{1 \times 2} = 36$.

E27. Prove : $^{17}C_6 = {}^{16}C_5 + {}^{16}C_6$

Sol. $^{16}C_5 + {}^{16}C_6 = \dfrac{16!}{5!\,11!} + \dfrac{16!}{6!\,10!}$. Multiply the first fraction by $\dfrac{6}{6}$ and the second by $\dfrac{11}{11}$ to obtain the same denominator in both fractions, and then add:

$$^{16}C_5 + {}^{16}C_6 = \frac{6 \times 16!}{6 \times 5! \times 11!} + \frac{11 \times 16!}{6! \times 11 \times 10!} = \frac{6 \times 16!}{6! \times 11!} + \frac{11 \times 16!}{6! \times 11!}$$

$$= \frac{6 \times 16! + 11 \times 16!}{6! \times 11!} + \frac{(6+11) \times 16!}{6! \times 11!} = \frac{17 \times 16!}{6! \times 11!} = \frac{17!}{6! \times 11!} = {}^{17}C_6$$

E28. Prove Theorem $^{n+1}C_r = {}^nC_{r-1} + {}^nC_r$

(The technique in this proof is similar to that of the preceding problem)

Sol. $^nC_{r-1} + {}^nC_r = \dfrac{n!}{(r-1)! \times (n-r+1)!} + \dfrac{n!}{r! \times (n-r)!}$. To obtain the same denominator in both fractions, multiply the first fraction by $\dfrac{r}{r}$ and the second fraction by $\dfrac{n-r+1}{n-r+1}$.

Hence

$$^nC_{r-1} + {}^nC_r = \frac{r \times n!}{r \times (r-1)! \times (n-r+1)!} + \frac{(n-r+1) \times n!}{r! \times (n-r+1) \times (n-r)!} =$$

$$\frac{r.n!}{r!.(n-r+1)!} + \frac{(n-r+1).n!}{r!(n-r+1)!} = \frac{r.n! + (n-r+1).n!}{r!(n-r+1)!} = \frac{[r+(n-r+1)].n!}{r!(n-r+1)!}$$

$$= \frac{(n+1)n!}{r!(n-r+1)!} = \frac{(n+1)!}{r!(n-r+1)!} = {}^{n+1}C_r.$$

E29. Assume there are three men and five women at a party. Show that if these people are lined up in a row, at least two women will be next to each other.

Notes / Rough Work

Sol. Consider the case where the men are placed so that no two men are next to each other and not at either end of the line. In this case, the three men generate four potential locations (pigeonholes) in which to place women (at either end of the line and two locations between men within the line). Since there are five women (pigeons), at least one slot will contain two women who must, therefore, be next to each other. If the men are allowed to be placed next to each other or at the end of the line, there are even fewer pigeonholes and, once again, at least two women will have to be placed next to each other.

E30. Find the minimum number of students needed to guarantee that five of them belong to the same class (Std. IX, X, XI, XII).

Sol. Here the n = 4 classes are the pigeonholes and k + 1 = 5 so k = 4. Thus among any kn + 1 = 17 students (pigeons), five of them belong to the same class.

E31. Simplify: (a) $\frac{(n+1)!}{n!}$; (b) $\frac{n!}{(n-2)!}$; (c) $\frac{(n-1)!}{(n+2)!}$; (d) $\frac{(n-r+1)!}{(n-r-1)!}$.

Sol. (a) n + 1; (b) $n(n-1) = n^2 - n$;
(c) 1/[n(n + 1) (n + 2)]; (d) (n –r) (n – r + 1).

Mini Revision Test # 01

1. There are 10 buses running between Delhi and Agra. In how many ways can a man go from Delhi to Agra and return by a different bus?
2. There are three routes: air, rail and road for going from Madras to Hyderabad. But from Hyderabad to Vikarabad, there are two routes, rail and road. How many kinds of routes are there from Madras to Vikarabad via Hyderabad?
3. In a text-book on Mathematics there are 3 exercises A, B, C consisting of 12, 18 and 9 questions respectively. In how many ways can three questions be selected choosing one from each exercise.
4. In a school, there are four sections of 40 students each for each class, in XI standard. In how many ways can a set of 4 student representatives be chosen, one from each section?
5. In how many ways can a vowel, a consonant and a digit be chosen out of the 26 letters of the English alphabet and the 10 digits?
6. In a class there are 27 boys and 14 girls. The teacher wants to select 1 boy and 1 girl to represent the class in a function. In how many ways can the teacher make this selection?
7. Given A = {2, 3, 5 } and B = { 0, 1 }. Find the number of different ordered pairs in which the first entry is an element of A and the second is an element of B.
8. How many arithmetic progressions with 10 terms are there whose first term is in the set {1, 2, 3} and whose common difference is in the set (2, 3, 4}?
9. How many 4-digit numbers are there with distinct digits?
10. How many 3-digit numbers of distinct digits can be formed from the digits 2, 3, 7, 8, 9?

Notes / Rough Work

Challenge Problems # 01

1. A class comprises 4 girls and 5 boys. Every Sunday 6 Students go on an excursion, a different group being sent each week. During every excursion, each girl in the group is given a toy by the accompanying teacher. After all the possible groups of 6 have gone twice, the total number of toys the girls have got is

(1) 56 (2) 112 (3) 224 (4) 448

2. In a colony the houses are numbered in a particular way. The first house has been numbered 1!. Second house has been numbered 1! + 2!. Similarly nth house is numbered 1! + 2! + 3! + ... + n!. Nikhilesh decided to buy the house in this colony. But his wife has one condition that the number of house should be square of natural number. If Nikhilesh is a blind follower of his wife how many options of houses does Nikhilesh have?

(1) 1 (2) 2 (3) 3 (4) More than 3

3. A five-digit number is formed using digits 1,3,5,7 and 9 without repetition . What is the sum of all such possible numbers?

(1) 6666600 (2) 6666660 (3) 6666666 (4) None of these

4. The number of subsets of the set $A = \{a_1, a_2, ..., a_n\}$ which contain even number of elements is

(1) 2^{n-1} (2) $2^n - 1$ (3) 2^{n-2} (4) 2^n

5. m men and w women are to be seated in a row so that no two women sit together. If m > w, then the number of ways in which they can be seated is

(1) $\frac{m!(m+1)!}{(m-w+1)!}$ (2) ${}^mC_m - w\ (m - w)!$

(3) m + wCm (m – w)! (4) None of these

Sequence

A succession of numbers formed and arranged in a definite order according to a certain definite rule is called a sequence.

The number occurring at the nth place of a sequence is called its **n^{th} term** or the general term, to be denoted by $\mathbf{t_n}$. A sequence is said to be finite or infinite according the number of distinct terms in it is finite or infinite.

Series

By adding the terms of a sequence, we obtain a series. A series is said to be finite or infinite depending on the number of terms added is finite or infinite.

SOME SPECIAL SEQUENCES

I. SUM OF FIRST n NATURAL NUMBERS

Let $S_n = 1 + 2 + 3 + ... + n$.
This is clearly an arithmetic series with $a = 1$, $d = 1$, $\ell = n$ and the number of terms = n.

$\therefore\ S_n = \left(\frac{n}{2}\right)(1+n) = \frac{n(n+1)}{2}$ Hence $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

II. SUM OF THE SQUARES OF FIRST n NATURAL NUMBERS

To find $\sum_{k=1}^{n} k^2$

Let $S_n = 1^2 + 2^2 + 3^2 + \dots + n^2$.
Consider the identity $x^3 - (x-1)^3 = 3x^2 - 3x + 1$ ... (i)
Putting x = 1, 2, 3, ... , (n – 1), n successively, we get:
$1^3 - 0^3 = 3.1^2 - 3.1 + 1;$
$2^3 - 1^3 = 3.2^2 - 3.2 + 1;$
$3^3 - 2^3 = 3.3^2 - 3.3 + 1;$
... ...
$(n-1)^3 - (n-2)^3 = 3.(n-1)^2 - 3.(n-1) + 1;$
$n^3 - (n-1)^3 = 3.n^2 - 3.n + 1.$
Adding columnwise, we obtain :
$n^3 = 3.[1^2 + 2^2 + 3^2 + \dots + (n-1)^2 + n^2]$
$-3.[1 + 2 + 3 + \dots + (n-1) + n] + [1 + 1 + \dots \text{ to n terms}]$

$$= \left[3.\left(\sum_{k=1}^{n} k^2\right) - 3.\left(\sum_{k=1}^{n} k\right) + n\right]$$

$$\therefore\ 3.\left(\sum_{k=1}^{n} k^2\right) = n^3 + 3\left(\sum_{k=1}^{n} k\right) - n$$

$$= n^3 + \frac{3n(n+1)}{2} - n\left[\because \sum_{k=1}^{n} k = \frac{n(n+1)}{2}\right] = \therefore \left(\sum_{k=1}^{n} k^2\right) = \frac{n(n+1)(2n+1)}{6}.$$

III. SUM OF THE CUBES OF FIRST n NATURAL NUMBERS

Notes / Rough Work

To find $\sum_{k=1}^{n} k^3$

Let $S_n = 1^3 + 2^3 + 3^3 + 4^3 + ... + n^3$

Consider the identity

$x^4 - (x-1)^4 = 4x^3 - 6x^2 + 4x - 1$ ...(i)

Putting x = 1, 2, 3, ... , (n – 1), n successively, we get:

$1^4 - 0^4 = 4 \times 1^3 - 6 \times 1^2 + 4 \times 1 - 1;$

$2^4 - 1^4 = 4 \times 2^3 - 6 \times 2^2 + 4 \times 2 - 1;$

$3^4 - 2^4 = 4 \times 3^3 - 6 \times 3^2 + 4 \times 3 - 1;$

... ...

$(n-1)^4 - (n-2)^4 = 4 \times (n-1)^3 - 6 \times (n-1)^2 + 4 \times (n-1) - 1$

$n^4 - (n-1)^4 = 4 \,.\, n^3 - 6 \,.\, n^2 + 4 \,.\, n - 1$

Adding columnwise, we obtain

$$n^4 = [4(1^3 + 2^3 + 3^3 + ... + n^3) - 6(1^2 + 2^2 + 3^2 + ... + n^2) + 4(1 + 2 + 3 + ... + n) - (1 + 1 + ... \text{+to n terms})]$$

$$= \left[4\left(\sum_{k=1}^{n} k^3\right) - 6\left(\sum_{k=1}^{n} k^2\right) + 4\left(\sum_{k=1}^{n} k\right) - n\right]$$

$$\therefore 4\left(\sum_{k=1}^{n} k^3\right) = \left[n^4 + 6\left(\sum_{k=1}^{n} k^2\right) - 4\left(\sum_{k=1}^{n} k\right) + n\right]$$

$$\left[n^4 + 6 \,.\, \frac{n(n+1)(2n+1)}{6} - 4 \,.\, \frac{n(n+1)}{2} + n\right]$$

$= n[n^3 + (n+1)(2n+1) - 2(n+1) + 1]$

$= n(n^3 + 2n^2 + n) = n^2(n+1)^2.$

$$\therefore \left(\sum_{k=1}^{n} k^3\right) = \frac{n^2(n+1)^2}{4} = \left[\frac{n(n+1)}{2}\right]^2 = \left(\sum_{k=1}^{n} k\right)^2$$

IV. SUM OF CUBES OF FIRST n ODD NATURAL NUMBERS

Let $S_n = 1^3 + 3^3 + 5^3 + ...$to n terms.

Clearly, the nth term of the series is $(2n-1)^3$.

$$\therefore S_n = \sum_{k=1}^{n} (8k^3 - 12k^2 + 6k - 1)$$

$$\left[8\left(\sum_{k=1}^{n} k^3\right) - 12 \,.\, \left(\sum_{k=1}^{n} k^2\right) + 6\left(\sum_{k=1}^{n} k\right) - n\right] \quad [\because 1 + 1 + ... \text{ to n terms} = n]$$

$= 8 \,.\, 1/4 n^2 (n+1)^2 - 12 \,.\, 1/6\, n(n+1)(2n+1) + 6 \,.\, 1/2\, n(n+1) - n$

$= 2n^2 (n+1)^2 - 2n(n+1)(2n+1) + 3n(n+1) - n$

$= n(n+1)(2n^2 - 2n + 1) - n$

$= n[(n+1)(2n^2 - 2n + 1) - 1] = n(2n^3 - n)$

$= n^2(2n^2 - 1).$

E32. Find the sum of n terms of the series whose nth term is

(i) $n(n + 3)$ (ii) $(n^2 + 2^n)$.

Sol. (i) $S_n = \sum_{k=1}^{n} k^2 + 3K$ [$\because t_n = (n^2 + 3n)$]

$= \sum_{k=1}^{n} k^2 + 3\left(\sum_{k=1}^{n} k\right) = \frac{1}{6}$ n(n+1) (2n + 1) + 3 . $\frac{1}{2}$ n (n + 1)

$= \frac{n(n+1)}{6}$. [(2n + 1) + 9]

$= \frac{1}{3}$ n(n + 1)(n + 5).

(ii) $t_n = n^2 + 2^n$.

Putting n = 1, 2, 3, ..., n successively, we get

$t_1 = 1^2 + 2^1$;

$t_2 = 2^2 + 2^2$;

$t_3 = 3^2 + 2^3$;

... ...

$t_n = n^2 + 2^n$.

Adding columnwise, we get

$S_n = (t_1 + t_2 + ... + t_n) = (1^2 + 2^2 + ... + n^2) + (2 + 2^2 + 2^3 + ... + 2^n)$

$= \left(\sum_{k=1}^{n} k^2\right) + \left[\frac{2(2^n - 1)}{(2-1)}\right]$

$= \frac{1}{6}$ n (n + 1) (2n + 1) + 2 $(2^n - 1)$

E33. Sum the series : 3 × 8 + 6 × 11 + 9 × 14 + ... to n terms.

Sol. t_n = (nth term of 3, 6, 9, ...) × (nth term of 8, 11, 14, ...)

= [3 + (n – 1) × 3] × [8 + (n – 1) × 3] = (3n) (3n + 5)

= $(9n^2 + 15n)$

$\therefore S_n = \sum_{k=1}^{n} (9k^2 + 15k)$

$= 9.\left[\frac{1}{6}n(n+1)(2n+1)\right] + 15.\left[\frac{1}{2}n(n+1)\right]$

$= \frac{3}{2}$ n (n +1) [(2n + 1) + 5)]

= 3n (n + 1) (n + 3).

Notes / Rough Work

E34. Sum the series 1.2.3 + 2.3.4 + 3.4.5 + ... to n terms.

Notes / Rough Work

Sol. t_n = (nth term of 1,2, 3, ...) × (nth term of 2, 3, 4, ...) × (nth term of 3, 4, 5, ...)

$= n \times [2 + (n - 1).1] \times [3 + (n - 1) . 1]$

$= n(n + 1)(n + 2) - [n^3 + 3n^2 + 2n].$

$$\therefore S_n = \sum_{k=1}^{n}(k^3 + 3k^2 + 2k)$$

$$= \left(\sum_{k=1}^{n} k^3\right) + 3\left(\sum_{k=1}^{n} k^2\right) + 2\left(\sum_{k=1}^{n} k\right)$$

$$= \frac{1}{4} n^2 (n + 1)^2 + 3 \times \frac{1}{6} n (n + 1) (2n + 1) + 2 . \frac{1}{2} n (n + 1)$$

$$= \frac{1}{4} (n + 1)[n(n + 1) + 2(2n + 1) + 4]$$

$$= \frac{1}{4} n(n + 1)(n + 2)(n + 3) .$$

E35. Find the sum of n terms of the series $1^2 + 3^2 + 5^2 + \ldots\ldots$to n terms.

Sol. $t_n = [1 + (n - 1) \times 2]^2 = (2n - 1)^2 = (4n^2 - 4n + 1).$

$$= S_n = \sum_{k=1}^{n}(4k^2 - 4k + 1) = \left[4\left(\sum_{k=1}^{n} k^2\right) - 4\left(\sum_{k=1}^{n} k\right) + n\right]$$

$$4 . \frac{1}{6} n (n + 1) (2n + 1) - 4 . \frac{1}{2} n (n + 1) + n$$

$$= \frac{n}{3} [2 (n + 1) (2n + 1) - 6 (n + 1) + 3]$$

$$\frac{n}{3} (4n^2 - 1).$$

E36. If S_1, S_2, S_3 are the sums of first n natural numbers, their squares, their cubes respectively, show that $9S_2^2 = S_3(1 + 8S_1)$.

Sol. We have $S_1 = \frac{1}{2} n (n + 1)$; $S_2 = \frac{1}{6} n (n + 1) (2n + 1)$ and $S_3 = \frac{1}{4} n^2(n + 1)^2$.

$$\therefore 9S_2^2 = 9\left[\frac{n(n+1)(2n+1)}{6}\right]^2 = \frac{1}{4} n^2 (n+1)^2 (2n+1)^2$$

And, $S_3 (1 + 8S_1) = \frac{1}{4} n^2 (n + 1)^2 \times [1 + 8 . \frac{1}{2} n (n + 1)]$

$= \frac{1}{4} n^2(n + 1)^2 (2n + 1)^2$. Hence $9S_2^2 = S_3 (1 + 8S_1)$.

Notes / Rough Work

E37. Show that $\dfrac{1\times 2^2 + 2\times 3^2 + ... + n\times(n+1)^2}{1\times 2 + 2^2\times 3 + ... + n^2\times(n+1)} = \dfrac{3n+5}{3n+1}$

Sol. nth term of the $N^r = n(n+1)^2 = n^3 + 2n^2 + n$;
nth term of the $D^r = n^2(n+1) = n^3 + n^2$.

$$\therefore \text{ L.H.S. } \frac{\sum_{k=1}^{n}(k^3+2k^2+k)}{\sum_{k=1}^{n}(k^3+k^2)} = \frac{\sum_{k=1}^{n}k^3 + 2\sum_{k=1}^{n}k^2 + \sum_{k=1}^{n}k}{\sum_{k=1}^{n}k^3 + \sum_{k=1}^{n}k^2}$$

$$= \frac{\frac{1}{4}n^2(n+1)^2 + 2\times\frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1)}{\frac{1}{4}n^2(n+1)^2 + \frac{1}{6}n(n+1)(2n+1)} \text{ [on simplifying]}$$

$$= \frac{3n+5}{3n+1} = \text{R.H.S.}$$

Mini Revision Test # 02

1. If the nth term of a series is written as n(n + 1), find the sum of the n terms.

   (1) n^2 (2) $\dfrac{n(n+1)(n+2)}{3}$

   (3) $\dfrac{(n+1)(2n+1)}{3}$ (4) None of these

2. Write down the fifth terms of the sequence with $T_n = \dfrac{4n-1}{5n+3}$.
3. Evaluate the factorial 7! / 2!
4. The sum of 1 + 3 + 5 + 7 + ... upto 8 terms is
5. Find the sum of first 100 natural numbers.
6. Find the sum to 20 terms of the series $1^3 + 2^3 + 3^3 + ...$
7. Find the sum to 10 terms of the series 1.2.4 + 2.3.7 + 3.4.10 + ...
8. Find the sum to 10 terms of the series 2 + 4 + 6 + 8 + ...
9. Find the sum to 20 terms of the series $1^2 + 2^2 + 3^2 + ...$
10. A ball is dropped from a height of 81 metres. Each time is strikes the ground if rebounds to two – thirds the distance through which is last fell.

    Find the total distance that the ball travels when it reaches the top most position after the 4th bounce?